AN INTRODUCTION
TO THE
THEORY OF LIFE CONTINGENCIES

INTRODUCTION TO THE THEORY OF LIFE CONTINGENCIES

In order to follow the present international standard actuarial notation and usage, the following examples illustrate what changes should be made from the symbols in this book.

For	Q	use $_{\infty}q$
	$\vert_{n}Q$	use $_{n}q$
	$_{m}\vert_{n}Q$	use $_{m\vert n}q$
For	$\vert_{n}e_x$	use $_{n}e_x$ or $e_{x\overline{n}\vert}$
	$\vert_{n}a_x$	use $_{n}a_x$ or $a_{x\overline{n}\vert}$
	a_x	use $\ddot{a}_x$
	$\vert_{n}\mathrm{a}_x$	use $_{n}\ddot{a}_x$ or $\ddot{a}_{x\overline{n}\vert}$
For	$\vert_{n}A$	use $_{n}A$
For	$^{(r)}P_x^{(m)}$	use $P^{(m)}(A_x^{(r)})$
	$^{(\infty)}\bar{P}_x$	use $\bar{P}(\bar{A}_x)$
For	$\mathbf{N}_x$	use N_x
	$\mathbf{S}_x$	use S_x
	N_x	use N_{x+1}
	S_x	use S_{x+1}

Before using the N_x values from a published table, one should check that it is a "modern" table by testing $N_x - N_{x+1} = D_x$ (last two digits should be enough). In particular, if D_w is the last $D_x > 0$, then $N_w = D_w$ in a modern table.

In an "old" table—published generally before 1950—we find $N_x - N_{x+1} = D_{x+1}$ and $N_w = 0$. In such a table the modern value of $N_{20} = D_{20} + D_{21} + D_{22} + \ldots\ldots$ will be found opposite $x = 19$.

AN INTRODUCTION

TO THE

THEORY OF LIFE CONTINGENCIES

AN INTRODUCTION
TO THE
THEORY OF LIFE CONTINGENCIES

BY

M. A. MACKENZIE AND N. E. SHEPPARD

PRICE TWO DOLLARS AND FIFTY CENTS

THE UNIVERSITY OF TORONTO PRESS
1931

PRINTED IN CANADA

PREFACE

This little book must not be regarded as an exhaustive text book on the Theory of Life Contingencies. It was written for the use of the authors' classes in this University, and covers only so much of the theory as we have been able to bring before our classes during a course extending over one academic year.

We have learned that students find their greatest difficulty in understanding the notation—in attaching precise meanings to the symbols used and in stating the meanings of the identities in their own words. We have attempted to help the student in this direction and to encourage him to interpret the results of his work.

We have tried as far as possible to confine ourselves to the mathematical theory and to avoid references to "Office Practice".

Our thanks are due to B. T. Holmes, J. D. Burk, and G. de B. Robinson for reading the manuscript and for making valuable suggestions.

M. A. Mackenzie
N. E. Sheppard

University of Toronto,
November 1931.

CONTENTS

AN INTRODUCTION TO THE THEORY OF LIFE CONTINGENCIES

CHAPTER I

The Life Table

1. Suppose that we could trace a great many human lives of the same type from birthday to birthday, noting the number who attained each exact age and the part of that number who died before attaining the next age. For example, suppose that we had 100,000 lives aged exactly 50 and that of this 100,000 there were 1,600 who died before attaining age 51: we would be justified in deducing probabilities of death or survival of another life of the same type during the same life year. Of such another life we could say that the probability of surviving from age 50 to age 51 was .984 and the probability of dying within the year was .016.

Such probabilities of death and survival from age to age have been calculated for many types of lives—male and female, insured lives, pensioners, Englishmen, Americans, etc.

2. Defining q_x as the probability that a life aged exactly x will die before attaining age $x+1$, *i.e.*, will not die after age $x+1$, and p_x as the probability that a life aged exactly x will attain age $x+1$, *i.e.*, will not die before age $x+1$, we have $p_x+q_x=1$ for any value of x.

3. Suppose that the values of p_x and q_x for every age have been ascertained for a particular class of lives. We can make what is called a Life Table. Defining l_x as the number who attain exact age x out of l_0 born alive and d_x as the number who out of the same l_0 die between the ages of x and $x+1$, and knowing the values of p_x and q_x for all values of x, we start with the arbitrary radix (an assumed number for l_0).

For illustration we shall use the English Life Table No. 8—Males. In this table

l_0	=	1,000,000 and values of d_x and l_x follow,
d_0	=	$120{,}441 = l_0 \times (q_0 = .120441)$
l_1	=	$879{,}559 = l_0 - d_0 = l_0 \times p_0$
d_1	=	$30{,}115 = l_1 \times (q_1 = .034239)$
l_2	=	$849{,}444 = l_1 - d_1 = l_1 \times p_1$
d_2	=	$11{,}353 = l_2 \times (q_2 = .013365)$
l_3	=	$838{,}091 = l_2 - d_2 = l_2 \times p_2$
		etc. etc. etc.
l_{104}	=	$16 = l_{103} - d_{103} = l_{103} \times p_{103}$
d_{104}	=	$11 = l_{104} \times (q_{104} = .70399)$
l_{105}	=	$5 = l_{104} - d_{104} = l_{104} \times p_{104}$
d_{105}	=	$4 = l_{105} \times (q_{105} = .78631)$
l_{106}	=	$1 = l_{105} - d_{105} = l_{105} \times p_{105}$
d_{106}	=	$1 = l_{106} \times (q_{106} = .85859)$

4. Thus $l_x \times p_x = l_{x+1}$

$l_x \times q_x = d_x$, since q_x is the annual death rate or annual rate of mortality at age x

$$l_x - l_{x+1} = d_x$$

$$l_x - l_{x+n} = \sum_{r=0}^{n-1} d_{x+r}$$

and $\quad l_0 = d_0 + d_1 + d_2 + \ldots$ to the end of the table.

It must be clearly understood that no figure in either the l_x column or the d_x column has any meaning when taken by itself. Each value of l is a number of survivors from among a number of persons l_x who were alive at an earlier age x. Each value of d_x represents the number of deaths between ages x and $x+1$ out of a number l living at some previous age. If every value of l_x and d_x in any life table were to be multiplied by 100 or divided by 10, it would not alter the information given by that table.

5. The probability that a life aged x will be living n years hence is $\frac{l_{x+n}}{l_x}$. This is written ${}_np_x$. It is also the probability that a life aged x will die sometime after attaining age $x+n$ and hence is equal to $\frac{1}{l_x}[d_{x+n}+d_{x+n+1}+d_{x+n+2}$ . . . to the end of the table].

The probability that a life aged x will be dead n years hence is $\frac{d_x+d_{x+1}+\ \ldots\ +d_{x+n-1}}{l_x}=\frac{l_x-l_{x+n}}{l_x}=1-{}_np_x$.

The probability that a life aged x will die in the nth year from now is $\frac{d_{x+n-1}}{l_x}=\frac{l_{x+n-1}-l_{x+n}}{l_x}={}_{n-1}p_x-{}_np_x$.

Also ${}_{n+m}p_x=\frac{l_{x+n+m}}{l_x}=\frac{l_{x+n}}{l_x}\cdot\frac{l_{x+n+m}}{l_{x+n}}={}_np_x\cdot{}_mp_{x+n}$.

And ${}_np_x=p_x\cdot p_{x+1}\cdot p_{x+2}\ \ldots\ldots\ldots\ldots\ p_{x+n-1}$.

The probability that a life aged x will die in the $(n+1)$th year from now may be written as follows:

$$_{n|}q_x=\frac{d_{x+n}}{l_x}={}_np_x\cdot q_{x+n}=\frac{l_{x+n}-l_{x+n+1}}{l_x}={}_np_x-{}_{n+1}p_x.$$

The symbol q_x which is the annual rate of mortality at age x refers to the one year of life from age x to age $x+1$. The expression ${}_nq_x$ has no meaning.

$|_nQ_x$ or $Q^1_{x\,\overline{n}|}$ stands for the probability that a life aged x will die within the next n years and is equal to $1-{}_np_x$.

$_n|Q_x$ or $Q_{x\,\overline{n}|}^{\ \ 1}$ stands for the probability that a period of at least n years will elapse before a life aged x will fail, or that a life aged x will not die within the next n years, *i.e.*, will survive the next n years, and is therefore equal to ${}_np_x$.

$_n|_mQ_x$ stands for the probability that a life aged x will live n years but will die before attaining age $x+n+m$.

$$_n|_mQ_x={}_np_x\cdot|_mQ_{x+n}=\frac{l_{x+n}-l_{x+n+m}}{l_x}={}_np_x-{}_{n+m}p_x.$$

6. Suppose that we have a large population maintained in a stationary condition by l_0 births and l_0 deaths in each calendar year, that the births and deaths are evenly distributed throughout each year, and that this population has been in this condition for a period longer than any person can live.

l_x represents the number of persons in this population who, during any consecutive twelve months, attain exact age x out of the l_0 births that occurred during the year which began exactly x years before this twelve month period began. So that l_x is a number per annum.

In this stationary population at any moment it can be said that l_x persons attained age x during the past twelve months. Many of them are still living between the ages of x and $x+1$, but some of them have died. If they had all been exposed to the risk of death for a full year after age x, there would have been d_x deaths, but they have been so exposed for half a year on the average and therefore there have been $\frac{1}{2}d_x$ deaths. Consequently at any moment the number of persons in this population who are aged x last birthday and who are living between the ages x and $x+1$ will be $l_x - \frac{1}{2}d_x = \frac{1}{2}(l_x + l_{x+1})$. This is written L_x; and the total population aged x and upwards which is written T_x will be $L_x + L_{x+1} + L_{x+2} + \ldots\ldots$ to end of table.

7. Since l_0 births occur each year and a year is taken as the unit of time, the number of births that occur at any one instant will be $l_0.dt$, and the number of survivors of the $l_0.dt$ who are living $x+t$ years after birth will be $l_{x+t}.dt$. Therefore in the stationary population there will, on any date, be $L_x = \int_0^1 l_{x+t}.dt$ persons living aged x last birthday. These are the survivors at any one instant of the births that occurred between x and $x+1$ years ago.

The total population aged x and upwards is

$$T_x = \sum_{n=0}^{\infty} L_{x+n} = \int_0^{\infty} l_{x+t}.dt = \int_x^{\infty} l_t.dt, \text{ and } T_0 = \int_0^{\infty} l_x dx.$$

At the beginning of any period of twelve months L_x persons are living aged x last birthday; during this period l_x persons attain age x, and l_{x+1} persons attain age $x+1$; also at the end of this period there are L_x living aged x last birthday. Therefore the number of persons who die aged x last birthday during the twelve months must be $L_x+l_x-l_{x+1}-L_x=d_x$.

Assuming that these deaths are uniformly distributed in each life year from age x to age $x+1$, we get

$L_x=\frac{1}{2}(l_x+l_{x+1})=l_x-\frac{1}{2}d_x=l_{x+\frac{1}{2}}$, and

$T_x=\frac{1}{2}l_x+l_{x+1}+l_{x+2}+\ \ldots\ldots$ to the end of the table.

8. $q_x=\dfrac{l_x-l_{x+1}}{l_x}$ is the annual death rate at age x; also $\dfrac{12(l_x-l_{x+\frac{1}{12}})}{l_x}=\dfrac{l_x-l_{x+\frac{1}{12}}}{\frac{1}{12}\,l_x}$ is the annual death rate during the month from age x to age $x+\frac{1}{12}$; and $\dfrac{l_x-l_{x+h}}{hl_x}$ is the annual death rate during the short period from age x to age $x+h$, therefore $\lim\limits_{h\to 0}\dfrac{l_x-l_{x+h}}{hl_x}=\dfrac{-1}{l_x}\cdot\dfrac{d\,l_x}{dx}$ is the nominal annual rate of mortality at age x. This is called the "force of mortality" and is written μ_x so that $\mu_x=\dfrac{-1}{l_x}\cdot\dfrac{d}{dx}\,l_x=-\dfrac{d}{dx}\log l_x$.

9. Again, the effective annual decrease in l_x at age x is $l_x-l_{x+1}=d_x$. The effective annual rate of mortality at age x is therefore $\dfrac{d_x}{l_x}=q_x$. Remembering that a year is the unit of time, the nominal annual change in l_x at age x is $\dfrac{d}{dx}l_x$ and is negative. The nominal annual rate of mortality at age x is therefore $\dfrac{1}{l_x}\left(-\dfrac{d}{dx}l_x\right)$, so that we have

$$\mu_x=-\frac{1}{l_x}\,\frac{d}{dx}\,l_x=-\frac{d}{dx}\log l_x=\frac{d}{dx}\operatorname{colog} l_x.$$

Similarly the nominal annual change in L_x is $\dfrac{d}{dx}L_x$ and is negative: but $L_x=\int_0^1 l_{x+t}.dt$. Therefore $\frac{d}{dx}L_x=l_{x+1}-l_x=-d_x$. Hence the nominal annual rate of mortality among the L_x persons living between the ages of x and $x+1$ is $\frac{1}{L_x}\left(-\frac{d}{dx}L_x\right)$. This is called the "central death rate" and is written m_x where

$$m_x=\frac{-1}{L_x}\frac{d}{dx}L_x=-\frac{d}{dx}\log L_x=\frac{d_x}{L_x}$$

$$=\frac{d_x}{\frac{1}{2}(l_x+l_{x+1})}=\frac{d_x}{l_x-\frac{1}{2}d_x}=\frac{q_x}{1-\frac{1}{2}q_x}\text{ approximately.}$$

The number of deaths among the l_x that will take place during the moment dx is $l_x-l_{x+dx}=-dl_x=l_x\mu_x dx$. Therefore the probability that a life aged x will die at that moment is

$$\frac{l_x-l_{x+dx}}{l_x}=-\frac{dl_x}{l_x}=\mu_x dx.$$

Similarly the probability that a life aged x will die at age $x+t$ is

$$\frac{l_{x+t}}{l_x}\cdot\frac{l_{x+t}-l_{x+t+dt}}{l_{x+t}}={}_tp_x\cdot\mu_{x+t}\cdot dt.$$

Therefore $q_x=\int_0^1 {}_tp_x\mu_{x+t}dt$

and $\quad d_x=l_x\int_0^1 {}_tp_x\mu_{x+t}dt=\int_0^1 l_{x+t}\mu_{x+t}dt.$

10. Various suggestions have been made for a law of mortality, such as (i) $l_x=k(86-x)$, k being constant

(ii) $p_x=ar^x$, or $q_x=1-ar^x$, a and r being constants

(iii) $\mu_x=A+Bc^x$, A, B and c being constants.

The student should verify the following:

Under (i) $d_x=k$; $p_x=\frac{85-x}{86-x}$; $q_x=\frac{1}{86-x}$;

$$\mu_x = \frac{1}{86-x}; \ L_x = k(85\tfrac{1}{2} - x); \ T_x = \frac{k}{2}(86-x)^2.$$

Under (ii) $l_x = ka^x r^{\frac{x(x-1)}{2}}$; $d_x = ka^x r^{\frac{x(x-1)}{2}}(1-ar^x)$;

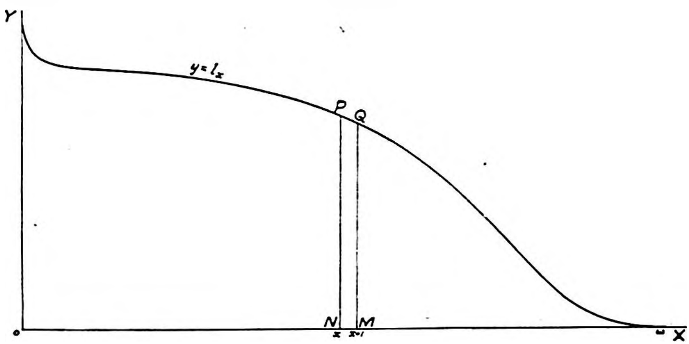
$$\mu_x = -\log a - \frac{2x-1}{2}\log r.$$

Under (iii) l_x would take the form $ks^x g^{c^x}$, where

$A = -\log s$ and $B = -\log g \,.\, \log c$,

so that ${}_tp_x = s^t g^{c^x(c^t-1)}$.

11. Consider the curve $y = l_x$, where the unit of length along the x-axis represents a year of life and a smaller unit of length along the y-axis represents one life, so that $y_0 = l_0$.



In this curve, PN represents l_x

QM represents l_{x+1}

so that $(PN - QM)$ represents d_x.

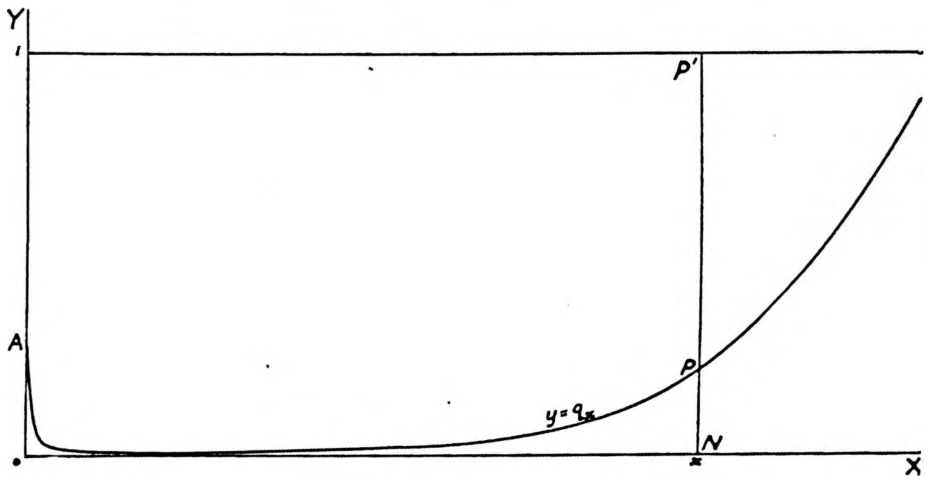
$L_x = \int_0^1 l_{x+t}\,dt =$ the area under the curve between PN and QM, which represents the number living in a stationary population between the ages of x and $x+1$, or the life-time in years to be enjoyed between these ages by l_x persons.

$T_x = \int_0^{\infty} l_{x+t}dt$ = the area under the curve and to the right of PN, which represents the number living in a stationary population aged x and upwards, or the life-time in years to be enjoyed after age x by l_x persons.

T_0 = the total area under the curve, which represents the total number living in the stationary population, or the total life-time after birth of the l_0 persons born in any given year.

Therefore, $\dfrac{T_x}{l_x}$ is the average future life time of a man aged x.

12. Consider the curve, $y = q_x$, where the unit of length along the x-axis represents one life year, but the length $O1$ on the Y axis represents unity, so that any shorter length must be a proper fraction and can represent the probability of death within one year.



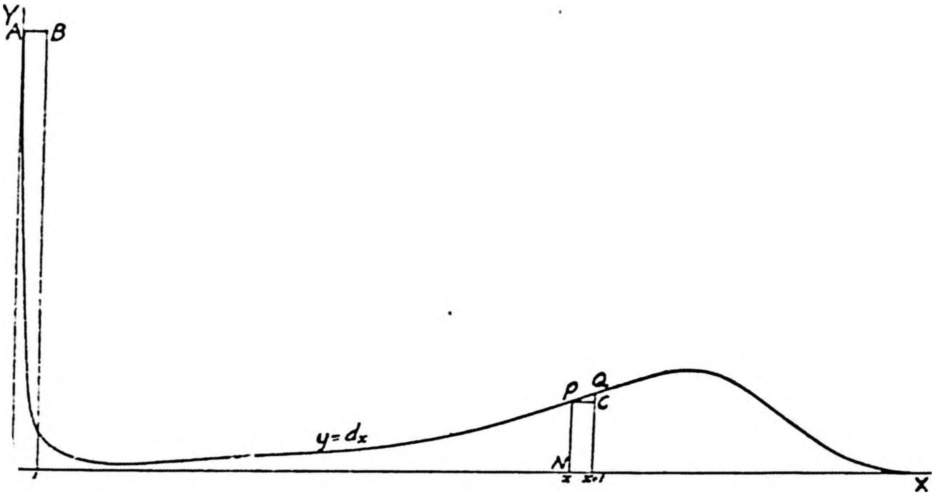
In this curve, $y_0 = q_0 \quad = OA$

$x \quad = ON$

$q_x \quad = PN$

and $\quad p_x \quad = P'P = 1 - q_x$.

13. Again, consider the curve $y = d_x = l_x q_x$, where the unit of length along the x-axis represents a year and a smaller unit of length along the y-axis represents one death.



this curve, $y_0=d_0=OA$, or the area of the rectangle OB, since $=1$.

$x=ON$

$d_x=PN$, or the rectangle NC since $PC=1$.

$_x=$ the sum of all the ordinates at unit intervals beginning W and going to the end of the curve, or the area made up m of all the rectangles to the right of PN since the width f these rectangles is unity.

CHAPTER II

Probabilities of Death and Survival

1. Starting with the values of q_x, we have in the first chapter deduced the columns l_x and d_x of the Life Table. We are now in a position to find the values of more complicated probabilities.

The symbol (x) is to be read as "a life now aged exactly x."

We have defined ${}_np_x = \frac{l_{x+n}}{l_x}$ as the probability that (x) would live at least n years longer, or that such a life would die at an age higher than $x+n$.

Also, ${}_{n|}Q_x = Q^1_{x\,\overline{n|}} = 1 - {}_np_x = \frac{l_x - l_{x+n}}{l_x}$ has been defined as the probability that (x) will die during the next n years or the probability that (x) will fail before the period of n years has elapsed, or the probability that (x) will not live n years.

2. Consider two lives (x) and (y) and assume that the death or survival of the one will not affect the death or survival of the other. The product $({}_np_x + {}_{n|}Q_x)\ ({}_np_y + {}_{n|}Q_y)$, which is 1×1, can be expanded as,

$${}_np_x \cdot {}_np_y + {}_np_x \cdot {}_{n|}Q_y + {}_np_y \cdot {}_{n|}Q_x + {}_{n|}Q_x \cdot {}_{n|}Q_y.$$

We proceed to examine the meaning of each of these four terms:

(1) Both (x) and (y) may survive n years, the probability being ${}_np_x \cdot {}_np_y$ which is written ${}_np_{xy}$ and which $= 1 - {}_{n|}Q_{xy} =$ the complement of the probability that the pair will be broken during the next n years.

Or $1 - Q_{\overset{1}{xy}\,\overline{n|}} =$ the complement of the probability that the pair will be broken first, before the n years are ended.

Or $Q_{xy\,\overset{1}{\overline{n|}}} =$ the probability that the n years will elapse before the pair is broken.

(2) (x) may survive the n years and (y) fail to do so, the probability being,

$$ {}_np_x \cdot |_nQ_y = {}_np_x \cdot Q^1_{y\,\overline{n|}} = {}_np_x(1-{}_np_y) = {}_np_x - {}_np_{xy} $$

= the probability that (x) will survive n years less the probability that the pair does so.

Or $$ |_nQ_y - |_nQ_x \cdot |_nQ_y = Q^1_{y\,\overline{n|}} - Q^1_{x\,\overline{n|}} \cdot Q^1_{y\,\overline{n|}} $$

= the probability that (y) dies within the period less the probability that both of them do so.

Or $$ |_nQ_y - |_nQ_{\overline{xy}} = Q^1_{y\,\overline{n|}} - Q^{\,1}_{\overline{xy}\,\overline{n|}} $$

= the probability that (y) will die within the period less the probability that the survivor of (x) and (y) will so die.

Or $Q^{\;\;2}_{x\underset{1}{y}\,\overline{n|}}$ = the probability that of the three statuses, the life of x, the life of y, and the period of n years certain, the life of y will fail first and the period of n years elapse second.

(3) (x) may die within the period and (y) survive it, the probabilities being as in (2) interchanging x and y.

(4) Both (x) and (y) may die within the period, the probability being

$$ |_nQ_x \cdot |_nQ_y = (1-{}_np_x)\,(1-{}_np_y) = (1-{}_np_x) - {}_np_y(1-{}_np_x) $$

= the probability that (x) will die within the n years less the probability that he will be the only one to do so.

Or $Q^{\,1}_{\overline{xy}\,\overline{n|}}$ = the probability that the survivor will die within the period.

Or $Q^{2}_{\underset{1}{x}y\,\overline{n|}} + Q^{\;\;2}_{x\underset{1}{y}\,\overline{n|}}$ which is the probability that (x) will die before the n years pass but not first of the three statuses plus the probability that (y) will die before the n years pass but not first of the three statuses.

Or $Q^1_{x\,\overline{n|}} - Q^1_{xy\,\overline{n|}} + Q^1_{y\,\overline{n|}} - Q_{x\overset{1}{y}\,\overline{n|}}$

$$ = Q^1_{x\,\overline{n|}} + Q^1_{y\,\overline{n|}} - Q^{\,1}_{\overline{xy}\,\overline{n|}} = |_nQ_x + |_nQ_y - |_nQ_{xy} $$

$$ = 1 - {}_np_x + 1 - {}_np_y - (1 - {}_np_{xy}) = 1 - ({}_np_x + {}_np_y - {}_np_{xy}) $$

$= 1 - {}_np_{\overline{xy}}$ = the complement of the probability that the survivor will live n years.

It should be noted that

$$ {}_{n}p_{\overline{xy}} = {}_{n}p_{xy} + {}_{n}p_x(1 - {}_{n}p_y) + {}_{n}p_y(1 - {}_{n}p_x) = {}_{n}p_x + {}_{n}p_y - {}_{n}p_{xy}. $$

3. Again, the probability that (x) will die in the nth year is

$$ {}_{n-1|}q_x = \frac{d_{x+n-1}}{l_x} = {}_{n-1}p_x \cdot q_{x+n-1} = {}_{n-1}p_x - {}_{n}p_x. $$

The probability that (x) will not die in the nth year is

$$ 1 - {}_{n-1|}q_x = \frac{l_x - d_{x+n-1}}{l_x} = (1 - {}_{n-1}p_x) + {}_{n}p_x = {}_{|n-1}Q_x + {}_{n}p_x, $$

which is the probability that (x) dies either during the first $(n-1)$ years or after n years.

The product

$$ \left[{}_{n-1|}q_x + (1 - {}_{n-1|}q_x)\right]\left[{}_{n-1|}q_y + (1 - {}_{n-1|}q_y)\right] $$

is unity and can be expanded as

$$ {}_{n-1|}q_x \cdot {}_{n-1|}q_y + {}_{n-1|}q_x(1 - {}_{n-1|}q_y) + {}_{n-1|}q_y(1 - {}_{n-1|}q_x) $$
$$ + (1 - {}_{n-1|}q_x)(1 - {}_{n-1|}q_y). $$

The meaning of each of these four terms should be carefully noted.

(1) Both (x) and (y) may die in the nth year from now,
(2) (x) may die in the nth year and (y) in some other year.
(3) (y) may die in the nth year and (x) in some other year, and
(4) neither of them may die in the nth year.

Notice also that

$$ {}_{n-1|}q_x \cdot {}_{n-1|}q_y = {}_{n-1}p_{xy} \cdot q_{\overline{x+n-1:y+n-1}}. $$

4. If we have three lives, one aged x, one aged y and one aged z, note the following probabilities concerning the next n years.

(i) That all will survive n years: ${}_{n}p_x \cdot {}_{n}p_y \cdot {}_{n}p_z$ which is written ${}_{n}p_{xyz}$.

(ii) That none will survive n years:

$$ (1 - {}_{n}p_x)\ (1 - {}_{n}p_y)\ (1 - {}_{n}p_z). $$

(iii) That at least one death will occur: $1 - {}_{n}p_{xyz}$.

(iv) That at least one life will survive:

$$1-(1-{}_np_x)\ (1-{}_np_y)\ (1-{}_np_z).$$

(v) That not more than one will die:

$${}_np_{xy}+{}_np_{yz}+{}_np_{zx}-2{}_np_{xyz}.$$

(vi) That not more than two will die: the same as (iv).

(vii) That not more than one will live:

$$1-({}_np_{xy}+{}_np_{yz}+{}_np_{zx})+2{}_np_{xyz}.$$

(viii) That not more than two will live: the same as (iii).

(ix) That the survivor will live out the period:

$${}_np_{\overline{xyz}}=\text{the same as (iv) or (vi).}$$

(x) That the survivor will not live out the period:

$$1-{}_np_{\overline{xyz}}=\text{the same as (ii).}$$

The probability that (x) will die during the nth year from now is ${}_{n-1}p_x\cdot q_{x+n-1}={}_{n-1}p_x(1-p_{x+n-1})={}_{n-1}p_x-{}_np_x$, therefore the probability that (x) will die before the nth year begins, that (y) will die in the nth year, and (z) survive the n years is

$$(1-{}_{n-1}p_x)({}_{n-1}p_y-{}_np_y){}_np_z.$$

5. It is a great advantage to use symbols for well-defined cases in the solving of most problems involving probabilities. Each symbol as used above is a function of all definitely placed prefixes or suffixes that may be attached to it, so that each symbol when completed represents a numerical value, depending on the positions and numerical values of the prefixes or suffixes that it supports..

The numerical values of l_x and d_x remain the base from which numerical values of the symbols representing probabilities of life and death more complicated than simple q_x can be produced. Hence it is important that the student should be able to turn given symbols into different forms especially those forms which involve the l_x and d_x columns.

The student must use much care in the production of his symbols. The ordinary operations of elementary mathematics and his common sense should be at his command in order to analyse the component parts of the problems he has to tackle or to obtain numerical values. Prefixes and suffixes may not follow the law of indices! The following notation should be studied:

$$l_{xy}=l_x.l_y,$$

$$d_{xy}=l_{xy}-l_{x+1:y+1}=l_{xy}-(l_x-d_x)(l_y-d_y)=l_yd_x+l_xd_y-d_xd_y,$$

so that $q_{xy}=\dfrac{d_{xy}}{l_{xy}}=q_x+q_y-q_x.q_y=q_x+q_y-q_{\overline{xy}}$,

while, $p_{xy}=\dfrac{l_{x+1:y+1}}{l_{xy}}=p_x.p_y$,

but $$\mu_{x+t:y+t}=\frac{-1}{l_{x+t:y+t}}\frac{d}{dt}l_{x+t:y+t}=-\frac{d}{dt}\log l_{x+t:y+t}$$

$$=-\frac{d}{dt}(\log l_{x+t}+\log l_{y+t})$$

$$=\mu_{x+t}+\mu_{y+t}.$$

6. The probability that (x) will die before (y) is written Q^1_{xy}. Now (x) may die in the nth year and (y) live to the end of the nth year, or they may both die in the nth year, and in the latter case if we assume an equal distribution of deaths the probability that (x) will die before (y) is $\frac{1}{2}$.

Therefore, $$Q^1_{xy}=\sum_{n=1}^{\infty}\left[({}_{n-1}p_x-{}_np_x){}_np_y+\tfrac{1}{2}({}_{n-1}p_x-{}_np_x)({}_{n-1}p_y-{}_np_y)\right]$$

$$=\sum_{n=1}^{\infty}({}_{n-1}p_x-{}_np_x)({}_{n-1}p_y+{}_np_y)\tfrac{1}{2}$$

which may be written

$$\sum_{n=1}^{\infty}({}_{n-1}p_x-{}_np_x)_{n-\frac{1}{2}}p_y=\frac{1}{l_{xy}}\sum_{n=1}^{\infty}d_{x+n-1}l_{y+n-\frac{1}{2}}.$$

This represents the sum of the probabilities, for all integral values of n, that (x) will die in the nth year and (y) live to at least the middle of that year.

$$Q^2_{xy}=Q_{x\overset{1}{y}}=1-Q^1_{xy},\text{ since }Q^1_{xy}+Q_{x\overset{1}{y}}=1.$$

Similarly, $Q_{\overset{1}{x}yz} = \sum\limits_{n=1}^{\infty} ({}_{n-1}p_x - {}_np_x)\,{}_{n-\frac{1}{2}}p_{yz}.$

and $Q_{\overset{2}{x}\underset{1}{y}z} = Q_{\overset{1}{x}z} - Q_{\overset{1}{x}yz}$

so $Q_{\overset{2}{x}yz} = Q_{\overset{1}{x}z} + Q_{\overset{1}{x}y} - 2Q_{\overset{1}{x}yz}$

also $Q_{\overset{3}{x}\underset{1}{y}\underset{2}{z}} = Q_{x\underset{1}{y}\overset{2}{z}}$

and $Q_{\overset{3}{x}yz} = 1 - Q_{\overset{1}{x}yz} - Q_{\overset{2}{x}yz} = 1 - Q_{\overset{1}{x}\,\overline{yz}}.$

Notice also $Q_{\overset{1}{x}y} = \int_0^{\infty} {}_tp_{xy}\mu_{x+t}dt$

and $Q_{\overset{2}{x}\underset{1}{y}z} = \int_0^{\infty} {}_tp_{xz}(1 - {}_tp_y)\mu_{x+t}dt$

and $|_nQ_{xy} = \int_0^{n} {}_tp_{xy}(\mu_{x+t} + \mu_{y+t})dt.$

7. If we have m lives all of the same age x, the mth power of ${}_np_x$ gives the probability that all of the m lives will be living n years hence. We are again assuming that the death or survival of one will not affect the time of death of any of the others.

Writing p for ${}_np_x$ and q for $1 - {}_np_x$, we have:—

The probability that all will survive the n years is p^m.

The probability that one specified life will fail to survive, while all the $(m-1)$ others will survive the n years is $p^{m-1}q$.

The probability that one and only one but any one of the m lives will not survive is $mp^{m-1}.q$.

The probability that two specified lives will die and the remaining $(m-2)$ will survive the period is $p^{m-2}.q^2$.

The probability that two and only two but any two will die while the others survive the period is $\binom{m}{2}\ p^{m-2}.q^2$.

Consider the expansion of $(p+q)^m$. The term $\binom{m}{r}p^{m-r}.q^r$ gives the probability that exactly r but any r of the lives will fail to survive the period.

The probability that at least two of the lives will survive is

$$1 - q^m - mpq^{m-1}.$$

The probability that not more than one of them will die is

$$p^m + mp^{m-1}.q.$$

The probability of exactly r deaths is

$$\frac{m(m-1)\ (m-2)\ .\ .\ .\ (m-r+1)}{\underline{|r}}\,p^{m-r}q^r.$$

The probability of exactly $r+1$ deaths is

$$\frac{m(m-1)\ (m-2)\ .\ .\ .\ (m-r)}{\underline{|r+1}}\,p^{m-r-1}.q^{r+1}.$$

Therefore, the probability of $r+1$ deaths will be greater than the probability of r deaths

so long as $\dfrac{m-r}{r+1}\cdot\dfrac{q}{p}>1$,

so long as $(m-r)q>(r+1)(1-q)$,

so long as $(r+1)<(m+1)q$.

But r must be an integer. Therefore the most probable number of deaths during the next n years among the m persons each aged x will be the greatest integer in $(m+1)q=(m+1)\ (1-{}_np_x)$.

What we usually want however is not the most probable number of deaths, but the mean number to be expected. If we have a great many lives m in number all of the same age x the probable or mean number of deaths to be expected within the coming year is mq, which may be taken to the nearest integer.

8. To find the total future life-time of l_x persons now aged x, assuming a uniform distribution of deaths throughout each year.

d_x will live for $\frac{1}{2}d_x$ years in the aggregate,

d_{x+1} will live for $\frac{3}{2}d_{x+1}$ years in the aggregate,

d_{x+2} will live for $\frac{5}{2}d_{x+2}$ years in the aggregate,

. . . .

. . . .

d_{x+n} will live for $\dfrac{2n+1}{2}\,d_{x+n}$ years in the aggregate.

. . . .

Making a total of $\frac{1}{2}l_x+l_{x+1}+l_{x+2}+\ .\ .\ .\ =T_x$ years.

The share of each of the l_x persons in this total is

$$\tfrac{1}{2}+p_x+{}_2p_x+{}_3p_x+\ .\ .\ .\ \text{to end of table.}$$

This is called the "complete expectation of life" at age x and is written, $\mathring{e}_x=\dfrac{T_x}{l_x}=\frac{1}{2}+\sum\limits_{n=1}^{\infty}{}_np_x=\sum\limits_{n=0}^{\infty}{}_np_x-\frac{1}{2}.$

The "curtate expectation of life", written e_x, is the average number of completed years of future life after age x. Since the lives will on the average die in the middle of the year of death,

$$\mathring{e}_x=\tfrac{1}{2}+e_x.$$

The probability that (x) will die in the $(n+1)$th year from now after completing n full years following age x is ${}_{n|}q_x=\dfrac{d_{x+n}}{l_x}$.

Since the sum of these probabilities for all integral values of n is 1, the mean or expected number of completed years is

$$e_x=\sum_{n=1}^{\infty}{}_{n|}q_x\times n=\frac{1}{l_x}\sum_{n=1}^{\infty}nd_{x+n}$$

$$=\frac{1}{l_x}[d_{x+1}+2d_{x+2}+3d_{x+3}+\ .\ .\ .\ .\ .\ ]$$

$$=\frac{1}{l_x}[l_{x+1}+l_{x+2}+l_{x+3}+\ .\ .\ .\ .\ .\ ],\text{ as before.}$$

Again, of the l_x persons, l_{x+1} will survive the first year, l_{x+2} will survive the second year, and so on, so that there will be

l_{x+1} completed years lived during the first year,

l_{x+2} completed years lived during the second year,

l_{x+3} completed years lived during the third year,

. . . .

. . . .

The total number of completed years therefore will be $l_{x+1}+l_{x+2}+l_{x+3}+\ .\ .\ .$ to end of table, and the average will be

$$e_x=p_x+{}_2p_x+{}_3p_x+\ .\ .\ .\ .$$

Again, consider one of the l_x persons. He may live through the nth year in which case he will have lived through one additional year of life, the probability of his doing so being ${}_n p_x$. His curtate expectancy, therefore, is $e_x = \sum_{n=1}^{\infty} {}_n p_x \times 1$.

Also the expectancy of life during the next n years is

$$|_n e_x = e_{x\overline{n}|} = \sum_{r=1}^{n} {}_r p_x = e_x - {}_n p_x \, e_{x+n}.$$

And the deferred expectancy after n years is

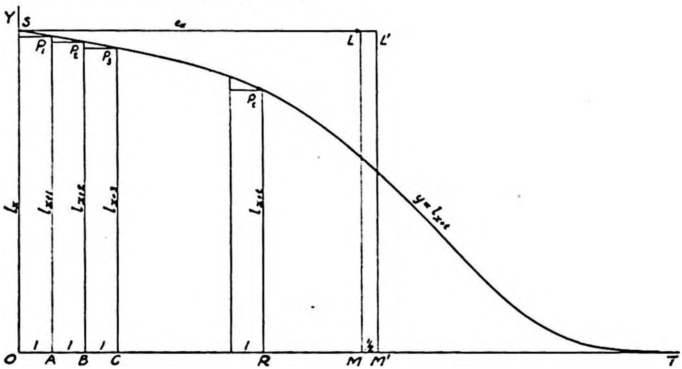
$$_{n|}e_x = e_x - e_{x\overline{n}|} = {}_n p_x \, e_{x+n}.$$

But $\mathring{e}_{x\overline{n}|} = \mathring{e}_x - {}_n p_x \, \mathring{e}_{x+n}$

$$= e_x - {}_n p_x \, e_{x+n} + \tfrac{1}{2}(1 - {}_n p_x)$$

$$= e_{x\overline{n}|} + \tfrac{1}{2}(1 - {}_n p_x).$$

9. Consider the curve $y = l_{x+t}$



In the above curve

$e_x \times l_x$ = the area of the rectangle OL.
= the sum of the areas of all the rectangles OP_1, AP_2, BP_3 . . . under the curve.

But $\mathring{e}_x = \int_0^{\infty} {}_t p_x \cdot dt$

$$\text{or } l_x \,.\, \mathring{e}_x = \int_0^\infty l_{x+t} \,.\, dt$$

= the total area under the curve
= the rectangle OL' on the assumption of a uniform distribution of deaths.

10. The expectancy of life at age x is not the period to which (x) has an even chance of living. For example in the English Life Table No. 8 the number living at age 20 is 793,435, and the number living at age 67 is 396,543. Thus (20) has approximately an even chance of living 47 years, but the expectancy of life at age 20 is 44.21 years.

Nor does the expectancy of life indicate the most probable year of death. Because $\mathring{e}_{20} = 44.21$, we cannot say that (20) is more likely to die between the ages of 64 and 65 than he is to die at any other age. In all life tables there is a maximum value of d_x, excluding the infantile ages. For example in the English Life Table No. 8—Males, every life at any age between the age of 2 and the age of 73 is more likely to die between the ages of 73 and 74 than at any other age because d_{x+n} has its maximum value when $x+n=73$ for each value of x from $x=2$ to $x=73$.

CHAPTER III

Assurances, Single Premiums

1. The value now of one payable for certain at the end of n years at interest rate i per annum is $(1+i)^{-n}=v^n=A_{\overline{n}|}$.

The value now of one payable at the end of n years if a definite event should happen is v^n multiplied by the probability that the event will happen.

For example,

The value now of one payable at the end of one year if (x) should be alive at the end of the year is

$$vp_x=v\frac{l_{x+1}}{l_x}=A_{x\,\overset{1}{\overline{1}|}}.$$

The value now of one payable at the end of one year if (x) should die within the year is

$$vq_x=v\frac{d_x}{l_x}=A^1_{x\,\overline{1}|}.$$

The value now of one payable at the end of n years if (x) should be alive at the end of the n years is

$$v^n{}_np_x=v^n\frac{l_{x+n}}{l_x}=A_{x\,\overset{1}{\overline{n}|}}.$$

The value now of one payable at the end of n years if (x) should die within the n years is

$$v^n\,.\,Q^1_{x\,\overline{n}|}=v^n(1-{}_np_x)=A_{\overline{n}|}-A_{x\,\overset{1}{\overline{n}|}}=A_{x\,\overset{2}{\overline{n}|}}.$$

The value now of one payable at the end of n years if (x) should die in the nth year from now is

$$v^n\cdot{}_{n-1}|\,q_x=v^n\,.\,\frac{d_{x+n-1}}{l_x}=v^n\frac{l_{x+n-1}}{l_x}-v^n\frac{l_{x+n}}{l_x}=vA_{x\,\overset{1}{\overline{n-1}|}}-A_{x\,\overset{1}{\overline{n}|}}.$$

2. The value now of one payable at the end of the year of the death of (x), *i.e.*, payable on the first anniversary of this date to which he does not survive, is written A_x

We have seen that (x), according to the life table, must fail during one and obviously can fail in only one of the next $\omega+1-x$ years where ω is the highest age for which a value appears in the l_x column of the life table. We have also seen that the terms of the expansion

$$1=\frac{d_x}{l_x}+\frac{d_{x+1}}{l_x}+\ldots+\frac{d_{x+n-1}}{l_x}+\ldots+\frac{d_\omega}{l_x}$$

represent the respective probabilities that (x) will fail in the first, second, . . . , nth, . . . and $(\omega+1-x)$th year from now. Further the value of one payable at the end of each of these years for certain is: v at the end of the first year, v^2 at the end of the second year, etc., up to $v^{\omega+1-x}$ at the end of the $(\omega+1-x)$th year.

Hence

$$A_x=v\frac{d_x}{l_x}+v^2\frac{d_{x+1}}{l_x}+\ldots+v^n\frac{d_{x+n-1}}{l_x}+\ldots+v^{\omega-x+1}\cdot\frac{d_\omega}{l_x}.$$

3. Again, consider l_x lives all aged exactly x and suppose one unit is to be paid at the end of the year of death of each of them, then

$1\times d_x$ will be required at the end of the first year.

$1\times d_{x+1}$ will be required at the end of the second year.

. . . .

$1\times d_{x+n-1}$ will be required at the end of the nth year.

. . . .

$1\times d_\omega$ will be required at the end of the $(\omega+1-x)$th year.

The total present value of all the payments to be made as the lives fail is

$$vd_x+v^2d_{x+1}+\ldots+v^n.d_{x+n-1}+\ldots+v^{\omega-x+1}.d_\omega,$$

and hence the average value required for a life now aged x is

$$A_x=\frac{1}{l_x}[vd_x+v^2d_{x+1}+\ldots+v^nd_{x+n-1}+\ldots+v^{\omega-x+1}d_\omega].$$

A_x is the value or single premium for a whole life assurance where the sum assured is one payable at the end of the year in which a life now aged exactly x fails.

4. The value or single premium for a term assurance where the sum assured is one to be paid at the end of the year in which (x) fails, should (x) die within n years, is written $A^{1}_{x\overline{n}|}$.

$$A^{1}_{x\overline{n}|}=v\frac{d_x}{l_x}+v^2\frac{d_{x+1}}{l_x}+\ldots+v^n\frac{d_{x+n-1}}{l_x}=\frac{1}{l_x}\sum_{r=1}^{n}v^r d_{x+r-1}$$

$$=\frac{1}{l_x}[vd_x+v^2d_{x+1}+\ldots+v^{\omega-x+1}d_\omega]$$

$$-\frac{1}{l_x}[v^{n+1}d_{x+n}+v^{n+2}d_{x+n+1}+\ldots+v^{\omega-x+1}d_\omega]$$

$$=A_x-v^n.\frac{l_{x+n}}{l_x}\cdot\frac{1}{l_{x+n}}[vd_{x+n}+v^2d_{x+n+1}+\ldots+v^{\omega-x-n+1}d_\omega]$$

$$=A_x-v^n{}_np_x.A_{x+n}$$

$$=A_x-A_{x\overline{n}|}^{\;\;1}.A_{x+n}.$$

5. Similarly, the value or single premium for a deferred life assurance where the sum assured is one to be paid at the end of the year in which (x) dies, provided (x) dies after attaining at least age $x+n$, is written ${}_{n|}A_x=A^{2}_{x\overline{n}|}$

$${}_{n|}A_x=\frac{1}{l_x}[v^{n+1}d_{x+n}+v^{n+2}d_{x+n+1}+\ldots+v^{\omega-x+1}d_\omega]=v^n{}_np_x.A_{x+n}.$$

Also $A^{2}_{x\overline{n}|}=A_x-A^{1}_{x\overline{n}|}$

so that $A_x=A^{1}_{x\overline{n}|}+{}_{n|}A_x=A^{1}_{x\overline{n}|}+A^{2}_{x\overline{n}|}$.

6. Again, $A_x=v\dfrac{d_x}{l_x}+v\dfrac{l_{x+1}}{l_x}\left[\dfrac{vd_{x+1}+v^2d_{x+2}+\ldots}{l_{x+1}}\right]=vq_x+vp_x.A_{x+1}$.

This can be written

$$\begin{aligned}A_x.(1+i)&=q_x\times1+p_x\times A_{x+1}\\&=A_{x+1}+q_x.(1-A_{x+1})\\&=1-p_x(1-A_{x+1})\end{aligned}$$

and further, as a means of obtaining A_{x+1} from A_x, we have

$$\begin{aligned}A_{x+1}&=A_x.\frac{1+i}{p_x}-\frac{q_x}{p_x}\\&=A_x.u_x-k_x,\text{ (Fackler's accumulation formula)}\end{aligned}$$

where $u_x = \frac{1+i}{p_x}$ and $k_x = \frac{q_x}{p_x}$, which are directly dependent on the rate of interest and the mortality basis.

7. The value or single premium for a pure endowment assurance where the sum assured is one to be paid at the end of the nth year from now provided (x) is then alive is, as above,

$$A_{x\overline{n}|}^{\;\;1} = v^n . \frac{l_{x+n}}{l_x} = v^n \cdot {}_np_x = v^n(1 - |_nQ_x) = A_{\overline{n}|} - v^n(1 - {}_np_x).$$

The value or single premium for an endowment assurance where the sum assured is one to be paid at the end of the policy year in which (x) dies, provided (x) dies before attaining age $x+n$, or to be paid at the end of n years provided x does not die before attaining age $x+n$ is

$$A_{x\overline{n}|} = v\frac{d_x}{l_x} + v^2\frac{d_{x+1}}{l_x} + \ . \ . \ . + v^n\frac{d_{x+n-1}}{l_x} + v^n\frac{l_{x+n}}{l_x} = A_{\overset{1}{x}\overline{n}|} + A_{x\overline{n}|}^{\;\;1}.$$

And $A_{\overline{x\overline{n}|}} = A^2_{x\overline{n}|} + A_{x\overline{n}|}^{\;\;2} = A_x + A_{\overline{n}|} - A_{x\overline{n}|}$.

Also $A_{x\overline{n}|}(1+i) = q_x + p_x A_{x+1:\overline{n-1}|}$.

Hence $A_{x+1:\overline{n-1}|} = A_{x\overline{n}|}u_x - k_x$.

8. The value or single premium for a whole life assurance where the sum assured is one payable at the end of the half year in which (x) dies is $A_x^{(2)}$.

Assume a uniform distribution of deaths in each policy year. In respect of the first year the probability that (x) dies in the first half of the year is $\frac{1}{2}\frac{d_x}{l_x}$, and the value of one payable at the end of half a year is $v^{\frac{1}{2}}$, therefore the value of one payable at the end of half a year if (x) should die within this half a year is $\frac{1}{2}v^{\frac{1}{2}}.\frac{d_x}{l_x}$, approximately. Similarly the value of one payable at the end of one year should (x) die in the second half of the first year is $\frac{1}{2}v\frac{d_x}{l_x}$.

Therefore.

$$A_x^{(2)} = \frac{v^{1/2}+v}{2} \cdot \frac{d_x}{l_x} \cdot + \frac{v^{3/2}+v^2}{2} \frac{d_{x+1}}{l_x} + \frac{v^{5/2}+v^3}{2} \cdot \frac{d_{x+2}}{l_x} + \dots$$

$$= \frac{v^{-\frac{1}{2}}+1}{2} \cdot \frac{1}{l_x} [vd_x + v^2 d_{x+1} + v^3 d_{x+2} + \dots]$$

$$= (1+\tfrac{1}{4}i)A_x \text{ approximately.}$$

Under a policy for which the single premium is $A_x^{(2)}$, (x) must die either in the first half or in the second half of the year of his death. If he should die in the first half of the year, 1 will be payable at the end of this half year which is approximately equivalent to a payment of $1+\frac{1}{2}i$ at the end of the year. If he should die in the second half of the year 1 will be payable at the end of the year. The probability of either event is assumed to be $\frac{1}{2}$, therefore the value at the end of the year of death is $\frac{1}{2}(1+\frac{1}{2}i)+\frac{1}{2}=1+\frac{1}{4}i$, approximately, or $A_x^{(2)}=A_x(1+\frac{1}{4}i)$, approximately.

Under a policy paid for by A_x, the sum assured will be payable on the average 6 months after the death. Under a policy paid for by $A_x^{(2)}$, the sum assured will be payable on the average 3 months after the death. Therefore the payment under $A_x^{(2)}$ will on the average be made 3 months earlier than the payment under A_x. Therefore, $A_x^{(2)}=A_x(1+\frac{1}{4}i)$ approximately; or (x) with a policy for 1 paid for by $A_x^{(2)}$ has the same property as (x) with a policy for $(1+\frac{1}{4}i)$ paid for by A_x per unit

9. On the same assumptions of

(1) simple interest, and

(2) a uniform distribution of deaths in each year

we have similarly,

$$A_x^{(4)} = (1+\tfrac{3}{8}i)A_x$$

$$A_x^{(m)} = \left(1+\frac{m-1}{2m}i\right)A_x$$

and

$$\bar{A}_x = (1+\tfrac{1}{2}i)A_x,$$

where $\bar{A}_x$ represents the value or single premium for a whole life assurance of one payable at the moment of the death of (x).

The true value of $\overline{A}_x$ is $\int_0^\infty v^t {}_tp_x\mu_{x+t}.dt.$

Similarly $\overline{A}^1_{x\,\overline{n|}} = \int_0^n v^t {}_tp_x\mu_{x+t}.dt.$

We have $A^{(2)}_{\substack{1\\x\,\overline{n|}}} = A^{(2)}_x - v^n {}_np_x A^{(2)}_{x+n}$

$$= (1+\tfrac{1}{4}i)A_x - v^n {}_np_x(1+\tfrac{1}{4}i)A_{x+n}$$

$$= (1+\tfrac{1}{4}i)A^1_{x\,\overline{n|}},$$

and $$A^{(m)}_{\substack{1\\x\,\overline{n|}}} = \left(1+\frac{m-1}{2m}i\right)A^1_{x\,\overline{n|}}.$$

But $$A^{(m)}_{x\,\overline{n|}} = A^{(m)}_{\substack{1\\x\,\overline{n|}}} + A_{x\,\overline{n|}}^{\;\;1}$$

$$= \left(1+\frac{m-1}{2m}i\right)A^1_{x\,\overline{n|}} + A_{x\,\overline{n|}}^{\;\;1}.$$

10. Consider $A_x = \dfrac{vd_x + v^2d_{x+1} + v^3d_{x+2} + \ldots\ldots}{l_x}.$

This may be regarded as the arithmetic mean of l_x quantities, namely, d_x each of the value v, d_{x+1} each of the value v^2, etc.

Now, $v^{1+e_x} = v^{\dfrac{l_x+l_{x+1}+l_{x+2}+\ldots}{l_x}}$

$$= v^{\dfrac{d_x+2d_{x+1}+3d_{x+2}+\ldots}{l_x}}$$

$$= \left[v^{d_x}.(v^2)^{d_{x+1}}.(v^3)^{d_{x+2}}\ldots\right]^{\frac{1}{l_x}},$$

which also may be regarded as the geometric mean of the same l_x quantities.

Now the *A.M.* of any number of positive quantities, not all equal to each other, is greater than the *G.M.* of these quantities, therefore $A_x > v^{1+e_x}$, and $\overline{A}_x > v^{\mathring{e}_x}$, since $\overline{A}_x = A_x(1+i)^{\frac{1}{2}}$ and $\mathring{e}_x = \frac{1}{2} + e_x$ approximately.

11. Suppose that we have two lives, one aged x and the other aged y.

The value or single premium for a joint life assurance where the sum assured is one to be paid at the end of the year in which the pair is broken is written A_{xy}.

$$A_{xy} = \sum_{n=1}^{\infty} v^n ({}_{n-1}p_{xy} - {}_np_{xy})$$

$$= \frac{1}{l_{xy}} \left[v(l_{xy} - l_{x+1:y+1}) + v^2(l_{x+1:y+1} - l_{x+2:y+2}) \dots\dots \right]$$

$$= \frac{1}{l_{xy}} [vd_{xy} + v^2 d_{x+1:y+1} + \dots\dots].$$

12. The value or single premium for a joint life term assurance where the sum assured is one to be paid at the end of the year in which the pair is broken, if the breakage occurs during the term, is written $A_{\overset{1}{xy}\,\overline{n|}}$, where the term is n years

$$A_{\overset{1}{xy}\,\overline{n|}} = \frac{1}{l_{xy}} [vd_{xy} + v^2 d_{x+1:y+1} + \dots + v^n d_{x+n-1:y+n-1}]$$

$$= A_{xy} - v^n {}_np_{xy} A_{x+n:y+n}.$$

13. The value or single premium for a survivorship life assurance where the sum assured is one to be paid at the end of the year in which the last survivor of (x) and (y) dies is written $A_{\overline{xy}}$,

$$A_{\overline{xy}} = \sum_{n=1}^{\infty} v^n ({}_{n-1}p_{\overline{xy}} - {}_np_{\overline{xy}})$$

$$= \sum_{n=1}^{\infty} v^n \left[({}_{n-1}p_x + {}_{n-1}p_y - {}_{n-1}p_{xy}) - ({}_np_x + {}_np_y - {}_np_{xy}) \right]$$

$$= \sum_{n=1}^{\infty} \left[v^n({}_{n-1}p_x - {}_np_x) + v^n({}_{n-1}p_y - {}_np_y) - v^n({}_{n-1}p_{xy} - {}_np_{xy}) \right]$$

$$= A_x + A_y - A_{xy}.$$

CHAPTER IV

ANNUITIES

1. A certain man is aged exactly x to-day. It is his birthday. He is to receive one dollar on every future anniversary of his birthday to which he may survive. What is the value now of these future payments? We designate this value by a_x, the symbol for an immediate annuity on (x).

If he is living n years hence he will be then aged $x+n$ and will receive one dollar, the value of which to-day is $\$1\times v^n$. But he may not be living n years hence, in which case that dollar will not be paid. The probability that he will live to receive it is ${}_np_x$. Therefore the real value to-day of the dollar which he may possibly receive when he is aged $x+n$ is $\$1.\times v^n \times {}_np_x$.

Therefore $\quad a_x=\sum\limits_{n=1}^{\infty} v^n{}_np_x = vp_x+v^2\cdot{}_2p_x+v^3\cdot{}_3p_x+\ .\ .\ .\ .$

2. Again, suppose that we have l_x persons living at exact age x, and that each one of them is to receive a payment of one at the end of each year of his life, the value of all these future payments will be

$vl_{x+1}+v^2l_{x+2}+v^3l_{x+3}+\ .\ .\ .$ to end of table, and the share of any one of the l_x persons will be

$$a_x=\frac{1}{l_x}[vl_{x+1}+v^2l_{x+2}+v^3l_{x+3}+\ .\ .\ .\ ]$$

$$=vp_x+v^2\cdot{}_2p_x+v^3\cdot{}_3p_x+\ .\ .\ .\ ,\text{ as above.}$$

3. Or again, if the annuitant were to die during the first year he would receive nothing. If he were to die during the second year, the value now of the payments he would receive is $a_{\overline{1|}}$. If he were to die during the third year, the value now of his receipts is $a_{\overline{2|}}$, and so on. The probabilities of death during the first, second, third, etc., years are

$$\frac{d_x}{l_x}, \frac{d_{x+1}}{l_x}, \frac{d_{x+2}}{l_x} \ldots \text{ respectively}$$

$$\therefore a_x = o \times \frac{d_x}{l_x} + a_{\overline{1}|} \times \frac{d_{x+1}}{l_x} + a_{\overline{2}|} \times \frac{d_{x+2}}{l_x} + \ldots$$

$$= \frac{1}{l_x}\left[vd_{x+1} + (v+v^2)d_{x+2} + (v+v^2+v^3)d_{x+3} + \ldots\right]$$

$$= \frac{1}{l_x}[vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \ldots].$$

If the first payment were to be made at once the value of the annuity would be $1 + a_x = \mathrm{a}_x$. This is called an annuity-due.

If $i = 0$, so that $v^n = 1$ for all values of n, we have

$$e_x = p_x + {}_2p_x + {}_3p_x + \ldots\ldots$$

4. If an annuity on (x) is not to run for more than n years, so that the maximum number of payments that can be made is n, we have a temporary annuity

$$a_{x\overline{n}|} = vp_x + v^2 \cdot {}_2p_x + v^3 \cdot {}_3p_x + \ldots + v^n{}_np_x.$$

$$= [vp_x + v^2{}_2p_x + v^3{}_3p_x + \ldots \text{ to end of table}]$$

$$-[v^{n+1}{}_{n+1}p_x + v^{n+2}{}_{n+2}p_x + \ldots \text{ to end of table}]$$

$$= a_x - v^n{}_np_x[vp_{x+n} + v^2{}_2p_{x+n} + \ldots]$$

$$= a_x - v^n{}_np_x a_{x+n}.$$

Also $\mathrm{a}_{x\overline{n}|} = \mathrm{a}_x - v^n{}_np_x\mathrm{a}_{x+n} = 1 + a_{x\overline{n-1}|}$.

5. If no payments of an annuity on (x) are to be made for the first n years, so that the annual payments of one each will begin when the man is aged $x+n+1$, we have a deferred annuity

$${}_n|a_x = v^{n+1} \cdot {}_{n+1}p_x + v^{n+2} \cdot {}_{n+2}p_x + \ldots$$

$$= v^n{}_np_x[vp_{x+n} + v^2 \cdot {}_2p_{x+n} + \ldots]$$

$$= v^n{}_np_x a_{x+n}$$

$$= a_x - a_{x\overline{n}|} = \mathrm{a}_x - \mathrm{a}_{x\overline{n+1}|}.$$

Similarly for a deferred annuity-due,

$$_{n|}\mathrm{a}_x = v^n {}_n p_x \, \mathrm{a}_{x+n} = \mathrm{a}_x - \mathrm{a}_{x\overline{n|}} = a_x - a_{x\overline{n-1|}}.$$

The following diagram which is drawn on the assumption that (x) will die after n years may be helpful:

$\longleftarrow$ n years $\longrightarrow$ year of death $\downarrow$

	x	$x+1$	$x+2$				$x+n$						
$a_x =$		1	1	...	1	1	1	1	1	1	...	1	1
$\mathrm{a}_x =$	1	1	1	...	1	1	1	1	1	1	...	1	1
$a_{x\overline{n\vert}} =$		1	1	...	1	1	1						
$\mathrm{a}_{x\overline{n\vert}} =$	1	1	1	...	1	1							
$_{n\vert}a_x =$								1	1	1	...	1	1
$_{n\vert}\mathrm{a}_x =$							1	1	1	1	...	1	1

Note that $\mathrm{a}_x - a_x = 1$

$$\mathrm{a}_{x\overline{n|}} - a_{x\overline{n|}} = 1 - v^n {}_n p_x$$

$$_{n|}\mathrm{a}_x - {}_{n|}a_x = v^n {}_n p_x.$$

6. If (x) has a whole life annuity of 1 per annum and if he should survive one year to age $x+1$, he will then have a property worth $1+a_{x+1}$, therefore $a_x = vp_x(1+a_{x+1}) = vp_x \, \mathrm{a}_{x+1}$.

This may be written

$$a_x + ia_x = p_x(1+a_{x+1}) + q_x \times 0$$

$$= (1+a_{x+1}) - q_x(1+a_{x+1}),$$

$p_x(1+a_{x+1})$

or

$$l_x a_x(1+i) = l_{x+1}(1+a_{x+1}) + d_x \times 0$$

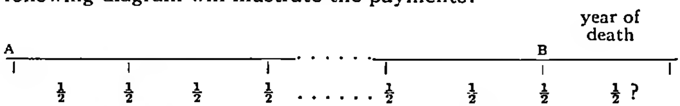
$$= l_x(1+a_{x+1}) - d_x(1+a_{x+1}).$$

The student should note the meanings of these identities.

7. Suppose that the payments under a life annuity on (x) were to be, not 1 each year, but 1 at the end of the first year, 2 at the end of the second year, 3 at the end of the third year and so on, the total value would be

$$\begin{aligned}(Ia)_x &= \sum_{n=1}^{\infty} nv^n{}_np_x = vp_x + 2v^2\cdot{}_2p_x + 3v^3\cdot{}_3p_x + \ldots \\ &= a_x + vp_x a_{x+1} + v^2{}_2p_x a_{x+2} + \ldots \\ &= \sum_{n=1}^{\infty} v^{n-1}{}_{n-1}p_x \cdot a_{x+n-1} \\ \text{or} \quad &= vp_x\cdot \mathrm{a}_{x+1} + v^2{}_2p_x\cdot \mathrm{a}_{x+2} + v^3{}_3p_x\cdot \mathrm{a}_{x+3} + \ldots \\ &= \sum_{n=1}^{\infty} v^n{}_np_x \mathrm{a}_{x+n}.\end{aligned}$$

8. Suppose that a life annuity on (x) of 1 per annum is to be paid twice a year: the symbol for it would be written $a_x^{(2)}$. The following diagram will illustrate the payments:



obviously $a_x^{(2)}$ is greater than a_x by the value of

(i) a possible $\frac{1}{2}$ that may be paid in the year of death

(ii) the interest on $\frac{1}{2}$ for half a year during each completed year of life.

On the assumption of a uniform distribution of deaths in each year, the annuitant is equally likely to die in either half of the last year, therefore the value at B of the possible last payment of $\frac{1}{2}$ is $\frac{1}{2}\times\frac{1}{2}\times v^{\frac{1}{2}} = \frac{1}{4}v^{\frac{1}{2}}$.

The value each year of the half year's interest on the $\frac{1}{2}$ is $\frac{1}{2}(1+i)^{\frac{1}{2}} - \frac{1}{2} = \frac{1}{4}i - \frac{1}{16}i^2 + -$etc.

A year's interest on the final $\frac{1}{4}v^{\frac{1}{2}}$ is $\frac{1}{4}(1+i)^{-\frac{1}{2}}\cdot i = \frac{1}{4}i - \frac{1}{8}i^2 + \ldots$

These are nearly equal and we can say that the sum of the values of (i) and (ii) above is approximately the value of an ultimate $\frac{1}{4}$ upon which interest will be paid until the $\frac{1}{4}$ is paid, which is the value of $\frac{1}{4}$ in cash.

Therefore $a_x^{(2)} = a_x + \frac{1}{4}$, approximately.

9. Similarly if a life annuity on (x) of 1 per annum is to be paid 4 times a year, its value, written $a_x^{(4)}$, will exceed the value of a_x by the value of

(i) an ultimate $\frac{1}{4}$, the probability of the payment of which is $\frac{3}{4}$ plus an ultimate $\frac{1}{4}$, the probability of the payment of which is $\frac{2}{4}$ plus an ultimate $\frac{1}{4}$, the probability of the payment of which is $\frac{1}{4}$, the total ultimate value being $\frac{1}{4}\times\frac{3}{4}+\frac{1}{4}\times\frac{2}{4}+\frac{1}{4}\times\frac{1}{4}=\frac{3}{8}$.

(ii) the interest during each year of life which is

$\frac{3}{4}$ of a year's interest on $\frac{1}{4}$ or a year's interest on $\frac{3}{16}$

$\frac{2}{4}$ of a year's interest on $\frac{1}{4}$ or a year's interest on $\frac{2}{16}$

$\frac{1}{4}$ of a year's interest on $\frac{1}{4}$ or a year's interest on $\frac{1}{16}$,

in all, a year's interest on $\frac{3}{8}$.

Therefore $a_x^{(4)}=a_x+\frac{3}{8}$, approximately.

Similarly it may be shown that,

$$a_x^{(m)}=a_x+\frac{m-1}{2m}\text{, approximately.}$$

If m be infinitely great, we obtain the value of an annuity of 1 per annum payable momently

$$\bar{a}_x=a_x+\tfrac{1}{2}\text{, approximately.}$$

Strictly, $\bar{a}_x=\int_0^\infty v^t{}_tp_x\cdot dt=\int_0^\infty \bar{a}_{\overline{t|}}\,{}_tp_x\cdot\mu_{x+t}dt.$

10. To find the value of $a_{x\overline{n|}}^{(m)}$ or $|_na_x^{(m)}$, we have

$$a_{x\overline{n|}}^{(m)}=|_na_x^{(m)}=a_x^{(m)}-v^n\cdot{}_np_x\cdot a_{x+n}^{(m)}$$

$$=\left(a_x+\frac{m-1}{2m}\right)-v^n{}_np_x\left(a_{x+n}+\frac{m-1}{2m}\right)\text{, approximately}$$

$$=a_{x\overline{n|}}+\frac{m-1}{2m}(1-v^n{}_np_x)$$

also $${}_n|a_x^{(m)}=v^n{}_np_x\left(a_{x+n}+\frac{m-1}{2m}\right)$$

and $${}_n|_ra_x^{(m)}=v^n{}_np_x\times|_ra_{x+n}^{(m)}=v^n{}_np_x\cdot a_{x+n:\overline{r|}}^{(m)}$$

$$=v^n{}_np_x\left[a_{x+n:\overline{r|}}+\frac{m-1}{2m}(1-v^r{}_rp_{x+n})\right].$$

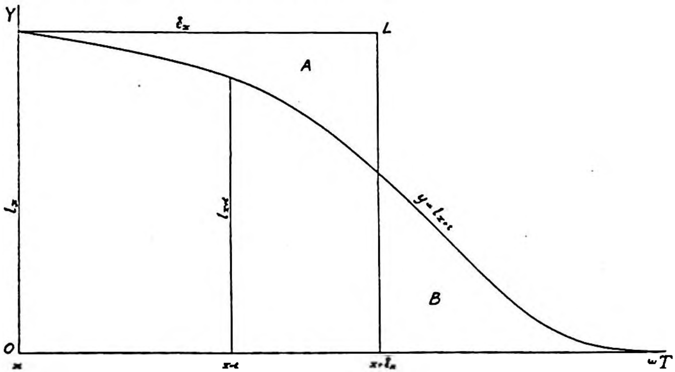
Also
$$\mathrm{a}_x^{(2)} = \tfrac{1}{2} + a_x^{(2)} = a_x + \tfrac{3}{4} = \mathrm{a}_x - \tfrac{1}{4}$$
$$\mathrm{a}_x^{(4)} = \tfrac{1}{4} + a_x^{(4)} = a_x + \tfrac{5}{8} = \mathrm{a}_x - \tfrac{3}{8}$$
$$\mathrm{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)} = a_x + \frac{m+1}{2m} = \mathrm{a}_x - \frac{m-1}{2m},$$

but
$$\bar{\mathrm{a}}_x = \bar{a}_x$$

and
$$\mathrm{a}_{x\overline{n|}}^{(m)} = \frac{1}{m} + a^{(m)}_{x:\overline{n-\frac{1}{m}|}}$$
$$= \frac{1}{m} + a_{x\overline{n|}}^{(m)} - v^n{}_n p_x \cdot \frac{1}{m}$$
$$= a_{x\overline{n|}} + \frac{m+1}{2m}(1 - v^n{}_n p_x) = \mathrm{a}_{x\overline{n|}} - \frac{m-1}{2m}(1 - v^n{}_n p_x),$$

or
$$\mathrm{a}_{x\overline{n|}}^{(m)} = \mathrm{a}_x^{(m)} - v^n{}_n p_x \cdot \mathrm{a}_{x+n}^{(m)}$$
$$= \left(\mathrm{a}_x - \frac{m-1}{2m}\right) - v^n{}_n p_x \left(\mathrm{a}_{x+n} - \frac{m-1}{2m}\right)$$
$$= \mathrm{a}_{x\overline{n|}} - \frac{m-1}{2m}(1 - v^n{}_n p_x).$$

11. Consider the curve $y = l_{x+t}$.



The total future life time of the l_x persons is $\int_0^\infty l_{x+t}.dt$ years.

The complete expectancy of life is $\frac{1}{l_x}\int_0^\infty l_{x+t}.dt = \int_0^\infty {}_tp_x.dt$ years.

The rectangle OL = the area under the curve and therefore the area A = the area B.

The value of $\bar{a}_x$, an annuity on (x) of 1 per annum payable momently, is represented by $\int_0^\infty v^t{}_tp_x dt$, where the payment at the end of time t is dt and the probability of survival to receive it is ${}_tp_x$.

Therefore $l_x\bar{a}_x = \int_0^\infty v^t l_{x+t}.dt$ = the sum of all the elements of the area under the curve such as $l_{x+t}.dt$ each multiplied by its own power of v.

But $l_x\bar{a}_{\overline{\mathring{e}_x}|} = l_x.\int_0^{\mathring{e}_x} v^t.dt$ = the sum of all the elements of the area of the rectangle OL such as $l_x.dt$ each multiplied by its own power of v.

Comparing $l_x.\int_0^\infty v^t{}_tp_x.dt$ and $l_x.\int_0^{\mathring{e}_x} v^t dt$, we see that so far as the part of the area which is common to both is concerned, they are identical. But although the area of A is equal to the area of B, the discount factor v has a smaller value with each element of B than it has with any element of A.

Therefore $\bar{a}_x$ must be less than $\bar{a}_{\overline{\mathring{e}_x}|}$.

12. Again, we have seen that

$$e_x = p_x + {}_2p_x + {}_3p_x + \ldots + {}_np_x + \ldots.$$

Consider the following expressions:

(1) $a_x = vp_x + v^2{}_2p_x + v^3{}_3p_x + \ldots + v^k{}_kp_x + \ldots + v^{\omega-x}{}_{\omega-x}p_x$,

where ω is the maximum age of the table, and k is the nearest integer to the value of e_x.

(2) $a_{\overline{e_x}|} = v + v^2 + v^3 + \ldots + v^k$, very nearly.

We can regard a_x as an annuity certain for $\omega-x$ years the successive annual payments being p_x, ${}_2p_x$, ${}_3p_x$ . . . decreasing in amount each year over the $\omega-x$ years, but the total sum to be received would be $p_x+{}_2p_x+\ldots=e_x$, which is the total sum to be received under $a_{\overline{e_x}|}$. So that, while the two annuities-certain, a_x and $a_{\overline{e_x}|}$, have the same total payments amounting to e_x, this total amount is to be become payable sooner under $a_{\overline{e_x}|}$ than under a_x.

Therefore $a_{\overline{e_x}|}$ must exceed a_x.

13. Suppose that we have two lives, one aged x and the other aged y, an annuity payable so long as both of them are alive would be represented by a_{xy}, where

$$a_{xy}=\frac{1}{l_{xy}}[vl_{x+1:y+1}+v^2l_{x+2:y+2}+\ldots\ldots]=\sum_{n=1}^{\infty}v^n{}_np_{xy}\,.$$

An annuity payable so long as there is at least one survivor of the pair (x) and (y) is represented by

$$a_{\overline{xy}}=a_x+a_y-a_{xy}.$$

An annuity payable so long as (y) may live after the death of (x) is represented by

$$a_x|_y=a_y-a_{xy}.$$

Also $\bar{a}_{xy}=\int_0^\infty v^t{}_tp_{xy}\,.\,dt=\int_0^\infty \bar{a}_{\overline{t}|}\,{}_tp_{xy}(\mu_{x+t}+\mu_{y+t})dt$

$$\bar{a}_{\overline{xy}}=\int_0^\infty v^t({}_tp_x+{}_tp_y-{}_tp_{xy})\,.\,dt,$$

and $\bar{a}_x|_y=\int_0^\infty v^t(1-{}_tp_x){}_tp_y\,.\,dt$

or $\bar{a}_x|_y=\int_0^\infty v^t{}_tp_{xy}\,.\,\mu_{x+t}\,.\,\bar{a}_{y+t}\,.\,dt.$

CHAPTER V

Connections between Assurance Single Premiums and Annuity Values

1. From the fundamental bases of interest and mortality, expressions have been obtained independently for the values of a few simple types of assurances and annuities. The values of these single premiums for assurances and annuities may be deduced from one another. The connecting links generally can be written in several forms, and the truth of each such form may be justified by a "verbal explanation".

The student should avail himself of every opportunity to consolidate the ideas that the symbols and the relations between them should convey to him by expressing these relations in his own words and so convincing himself of their truth.

2. If a loan of 1 is made for n years, the lender will receive interest each year and get the 1 back at the end of the nth year, or

$$1 = ia_{\overline{n}|} + v^n \quad \text{(i)}$$
$$1 = d\ddot{a}_{\overline{n}|} + v^n \quad \text{(ii)}$$

Similarly, a loan of 1 made to (x) until the end of the year of his death will produce interest during the period of the loan and a return of the capital at the end of that period, or

$$1 = ia_x + (1+i)A_x \quad \text{(i)}$$
$$1 = d\ddot{a}_x + A_x \quad \text{(ii)}$$

(i) is on the assumption that interest is paid at the end of each year,

(ii) is on the assumption that interest is paid at the beginning of each year.

From (i), in the case of the loan for the life of (x), we get

$$v = via_x + A_x$$

or

$$1 - d = da_x + A_x$$

or

$$1 = d\ddot{a}_x + A_x.$$

3. Consider,

$$A_{x\,\overset{1}{\overline{n|}}} = v^n\,{}_np_x = \sum_{r=1}^{n} v^r\,{}_rp_x - \sum_{r=1}^{n-1} v^r\,{}_rp_x = a_{x\,\overline{n|}} - a_{x:\overline{n-1|}}\,.$$

That is, the value now of one payable at the end of n years, if (x) should be alive at the end of n years, or the value of a pure endowment of 1 to (x) due at the end of n years, can be expressed as the difference in value between two temporary annuities of 1 each on the life (x). The first annuity to provide 1 per annum for a possible n years while the second annuity will withdraw 1 per annum for a possible $n-1$ years.

4. Again, $$A^{1}_{x\,\overline{n|}} = \sum_{r=1}^{n} v^r \frac{d_{x+r-1}}{l_x}$$

$$= \sum_{r=1}^{n} v^r \left(\frac{l_{x+r-1} - l_{x+r}}{l_x}\right)$$

$$= v\sum_{r=1}^{n} v^{r-1}\frac{l_{x+r-1}}{l_x} - \sum_{r=1}^{n} v^r \frac{l_{x+r}}{l_x}.$$

or $$A^{1}_{x\,\overline{n|}} = v\text{a}_{x\,\overline{n|}} - a_{x\,\overline{n|}}.$$

In words, we may say that the value now of one payable at the end of the year in which (x) dies, should (x) die within the next n years, is the difference in value between two annuities. The first annuity will provide 1 at the end of each year on which (x) enters for a possible n years, while the second annuity will withdraw 1 at the end of each year that he completes for a possible n years: the difference being the value of the 1 payable at the end of that year within the period which he may begin and not complete.

Also, $$A_{x\,\overline{n|}} = A_{x\,\overset{1}{\overline{n|}}} + A^{1}_{x\,\overline{n|}}$$

$$= a_{x\,\overline{n|}} - a_{x\,\overline{n-1|}} + v\text{a}_{x\,\overline{n|}} - a_{x\,\overline{n|}}$$

$$= v\text{a}_{x\,\overline{n|}} - a_{x\,\overline{n-1|}}\,.$$

This relation may also be stated in words in a similar manner so as to confirm its truth.

5. Also $A_x = \frac{1}{l_x}\,[vd_x + v^2 d_{x+1} + \ldots + v^n d_{x+n-1} + \ldots .]$

$$= \frac{1}{l_x}\left[v(l_x - l_{x+1}) + v^2(l_{x+1} - l_{x+2}) + \ldots + v^n(l_{x+n-1} - l_{x+n}) + \ldots\right]$$

$$= \frac{v}{l_x}\,[l_x + vl_{x+1} + \ldots + v^{n-1} l_{x+n-1} + \ldots]$$

$$- \frac{1}{l_x}\,[vl_{x+1} + v^2 l_{x+2} + \ldots + v^n l_{x+n} + \ldots .]$$

or $\quad A_x = v\mathrm{a}_x - a_x = (1-d)\,\mathrm{a}_x - a_x = 1 - d\,\mathrm{a}_x$

or $\quad 1 = d\,\mathrm{a}_x + A_x.$

The truth of each of these relations should be made evident in words.

6. Again, from first principles, we have

$$A_x = \frac{1}{l_x}\,[vd_x + v^2 d_{x+1} + \ldots + v^n d_{x+n-1} + \ldots]$$

$$= \frac{1}{l_x}\left[l_x - (d_x + d_{x+1} + \ldots + d_{x+n-1} + \ldots) + (vd_x + v^2 d_{x+1} + \ldots + v^n d_{x+n-1} + \ldots)\right]$$

$$= 1 - \frac{1}{l_x}\left[(1-v)d_x + (1-v^2)d_{x+1} + \ldots + (1-v^n)d_{x+n-1} + \ldots\right]$$

$$= 1 - \frac{d}{l_x}\left[d_x + (1+v)\,d_{x+1} + \ldots + (1 + v + v^2 + \ldots + v^{n-1})d_{x+n-1} + \ldots\right]$$

$$= 1 - \frac{d}{l_x}\,[l_x + vl_{x+1} + v^2 l_{x+2} + \ldots + v^n l_{x+n} + \ldots]$$

$$= 1 - d\mathrm{a}_x,$$

or $\quad 1 = d\mathrm{a}_x + A_x.$

7. By analogy,

$1 = ia_x + (1+i)\,A_x$, so $1 = j_{(m)} a_x^{(m)} + \left(1 + \frac{j_{(m)}}{m}\right) A_x^{(m)}$,

$1 = d\mathrm{a}_x + A_x$, so $1 = f_{(m)} \mathrm{a}_x^{(m)} + A_x^{(m)}$.

but each becomes $1=\delta\bar{a}_x+\bar{A}_x$ where,

$$(1+i)=\left(1+\frac{j_{(m)}}{m}\right)^m=\left(1-\frac{f_{(m)}}{m}\right)^{-m}=(1-d)^{-1}=e^{\delta}.$$

The corresponding statements in words are left as an exercise for the student.

8. Further, rewriting $A_x=1-d\mathrm{a}_x$, we have

$$\mathrm{a}_x=\frac{1-A_x}{d}=\frac{1}{d}-\frac{1}{d}\cdot A_x.$$

This result for a_x could have been deduced from first principles in a manner similar to that used to prove $A_x=1-d\mathrm{a}_x$.

Expressing the relation in words, we may say that the value now of an annuity-due of 1 per annum on the life of (x) is the difference in value between two perpetuities-due each of 1 per annum. The first perpetuity-due will provide 1 at the beginning of each year starting at once, while the second perpetuity-due will withdraw 1 each year starting at the beginning of the year following the death of (x).

Or we may say, since $1-A_x=d\mathrm{a}_x$, that $1-A_x$ represents the value now of an annuity-due of d per annum so long as (x) may live, and in proportion $\dfrac{1-A_x}{d}$ will represent the value now of an annuity-due of 1 per annum so long as (x) may live.

9. Similarly, we may write,

$$a_x=\frac{1-(1+i)A_x}{i}=\frac{1}{i}-\frac{1}{d}\cdot A_x,$$

also

$$a_x^{(m)}=\frac{1-\left(1+\dfrac{j_{(m)}}{m}\right)A_x^{(m)}}{j_{(m)}}=\frac{1}{j_{(m)}}-\frac{1}{f_{(m)}}\cdot A_x^{(m)},$$

$$\mathrm{a}_x^{(m)}=\frac{1-A_x^{(m)}}{f_{(m)}}=\frac{1}{f_{(m)}}-\frac{1}{f_{(m)}}\cdot A_x^{(m)},$$

and $$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} = \frac{1}{\delta} - \frac{1}{\delta}\cdot \bar{A}_x, \text{ where again}$$

$$(1+i)=(1-d)^{-1}=\left(1+\frac{j_{(m)}}{m}\right)^m=\left(1-\frac{f_{(m)}}{m}\right)^{-m}=e^{\delta}.$$

10. Corresponding expressions may be written for the case of Endowment Assurances, viz,,

$$A_{x\overline{n}|}=1-d\,\mathrm{a}_{x\overline{n}|}$$

or $$1=d\,\mathrm{a}_{x\overline{n}|}+A_{x\overline{n}|}$$

$$1=i\,a_{x\overline{n-1}|}+(1+i)A_{x\overline{n}|}=i\,a_{x\overline{n}|}+(1+i)A^1_{x\overline{n}|}+A_{x\overline{n}|}^{\ \ 1}$$

$$1=f_{(m)}\mathrm{a}^{(m)}_{x\overline{n}|}+A^{(m)}_{x\overline{n}|}$$

$$1=j_{(m)}a^{(m)}_{x:\overline{n-\frac{1}{m}}|}+\left(1+\frac{j_{(m)}}{m}\right)A^{(m)}_{x\overline{n}|}$$

$$=j_{(m)}\,a^{(m)}_{x\overline{n}|}+\left(1+\frac{j_{(m)}}{m}\right)A^{(m)\,1}_{x\overline{n}|}+A_{x\overline{n}|}^{\ \ 1}$$

and $$1=\delta\,\bar{a}_{x\overline{n}|}+\bar{A}_{x\overline{n}|}.$$

11. Solving each of the above for the annuity value, we have

$$\mathrm{a}_{x\overline{n}|}=\frac{1-A_{x\overline{n}|}}{d}=\frac{1}{d}-\frac{1}{d}\cdot A_{x\overline{n}|}$$

$$a_{x\overline{n-1}|}=\frac{1-(1+i)A_{x\overline{n}|}}{i}=\frac{1}{i}-\frac{1}{d}\cdot A_{x\overline{n}|}$$

$$a_{x\overline{n}|}=\frac{1}{i}-\frac{1}{d}\cdot A^1_{x\overline{n}|}-\frac{1}{i}\cdot A_{x\overline{n}|}^{\ \ 1}$$

$$\mathrm{a}^{(m)}_{x\overline{n}|}=\frac{1-A^{(m)}_{x\overline{n}|}}{f_{(m)}}=\frac{1}{f_{(m)}}-\frac{1}{f_{(m)}}\cdot A^{(m)}_{x\overline{n}|}$$

$$a^{(m)}_{x:\overline{n-\frac{1}{m}}|}=\frac{1-\left[1+\frac{j_{(m)}}{m}\right]A^{(m)}_{x\overline{n}|}}{j_{(m)}}=\frac{1}{j_{(m)}}-\frac{1}{f_{(m)}}\cdot A^{(m)}_{x\overline{n}|}$$

$$a_{x\,\overline{n}|}^{(m)}=\frac{1}{j_{(m)}}-\frac{1}{f_{(m)}}\cdot A_{\substack{1\\x\,\overline{n}|}}^{(m)}-\frac{1}{j_{(m)}}\cdot A_{x\,\overline{n}|}^{\;\;1}$$

$$\bar{a}_{x\overline{n}|}=\frac{1-\bar{A}_{x\overline{n}|}}{\delta}=\frac{1}{\delta}-\frac{1}{\delta}\cdot\bar{A}_{x\overline{n}|}.$$

12. It will be noticed, in the case of whole life and endowment assurance, that there is a very simple linear relation between the value of the assurance and the value of the corresponding annuity.

For example, we have $1=d(1+a_x)+A_x$

and $1=d(1+a_{x\,\overline{n-1}|})+A_{x\overline{n}|}$.

That is, if α represents the value of the assurance and β the value of the corresponding annuity, we have

$$1=d(1+\beta)+\alpha.$$

Since the range of β is limited to values lying between 0 and $\frac{1}{i}$ we can prepare tables, one for each rate of interest desired, showing the value of α corresponding to values of β beginning at 0 and rising, by differences of .01 for example, up to $\frac{1}{i}$. Such a table is called a Single Premium Conversion Table.

If we enter such a conversion table with the value a_x at a given rate of interest we arrive at the value A_x at the same rate of interest and conversely.

Also entering the same conversion table with the value $a_{x\,\overline{n-1}|}$ we arrive at the value $A_{x\overline{n}|}$.

13. Consider, again, the difference in value between $a_x^{(m)}$ and a_x

Due to the earlier payments made under $a_x^{(m)}$, the difference *each year* amounts to interest on $\frac{1}{m}$ for $\frac{m-1}{m}$ of a year

plus interest on $\frac{1}{m}$ for $\frac{m-2}{m}$ of a year

$$\text{plus interest on } \frac{1}{m} \text{ for } \frac{m-3}{m} \text{ of a year}$$

etc. etc. etc.

$$\text{plus interest on } \frac{1}{m} \text{ for } \frac{1}{m} \text{ of a year.}$$

The total value, assuming simple interest, amounts to a year's interest on $\frac{1}{m^2}\Big[(m-1)+(m-2)+(m-3)+\ \ldots\ +2+1\Big]=\frac{m-1}{2m}$.

Therefore the earlier payments during all the *completed* years will produce a difference worth $\frac{m-1}{2m}ia_x$ now.

If the life survives the first $\frac{1}{m}$ part of the year of death a payment of $\frac{1}{m}$ will be made under $a_x^{(m)}$ which would not be made under a_x. The probability of such a payment being $\frac{m-1}{m}$, its value at the end of the year of death is $\frac{m-1}{m}\cdot\frac{1}{m}\left(1+\frac{m-1}{m}i\right)$. Similarly at the end of the year of death the value of the second possible additional payment of $\frac{1}{m}$ is $\frac{m-2}{m}\cdot\frac{1}{m}\left(1+\frac{m-2}{m}i\right)$, and so on. The total value at the end of the year of death of all such possible payments is

$$\frac{1}{m^2}\Big[(m-1)+(m-2)+(m-3)+\ldots+2+1$$
$$+\frac{i}{m}\left(\overline{m-1}^2+\overline{m-2}^2+\ldots+2^2+1^2\right)\Big]$$

$$=\frac{1}{m^2}\left[\frac{m(m-1)}{2}+\frac{i}{m}\cdot\frac{(m-1)m(2m-1)}{6}\right]$$

$$=\frac{m-1}{2m}\left[1+\frac{2m-1}{3m}i\right].$$

The value now of these possible additional payments is

$$\frac{m-1}{2m}\left(1+\frac{2m-1}{3m}i\right)A_x$$

$=\frac{(m-1)}{2m}(1+i)A_x-\frac{m^2-1}{m^2}\cdot\frac{i}{6}\cdot A_x$, the last term of which is quite small.

Therefore, $a_x^{(m)}-a_x=\frac{m-1}{2m}\left[ia_x+(1+i)A_x\right]=\frac{m-1}{2m}$, approximately.

14. The annuity represented by a under which the last payment is made on the payment date preceding death is called a curtate annuity. If however a payment is to be made at death for the portion of the payment-period lived through before death, the annuity is called complete or apportionable and is written $\mathring{a}$.

Thus $\mathring{a}_x=a_x+\frac{1}{2}\overline{A}_x=a_x+\frac{1}{2}(1+\frac{1}{2}i)A_x$ approximately

and $\mathring{a}_x^{(m)}=a_x^{(m)}+\frac{1}{2m}\overline{A}_x=a_x^{(m)}+\frac{1}{2m}(1+\frac{1}{2}i)A_x$ approximately.

15. The connecting links for assurance and annuity symbols involving two lives would follow by the same methods as for one life.

$$\text{Thus } A_{xy}=\frac{1}{l_{xy}}\left[v(l_{xy}-l_{x+1:y+1})+v^2(l_{x+1:y+1}-l_{x+2:y+2})+\ldots\right]$$

$$=\frac{1}{l_{xy}}[vl_{xy}+v^2l_{x+1:y+1}+\ldots]$$

$$-\frac{1}{l_{xy}}[vl_{x+1:y+1}+v^2l_{x+2:y+2}+\ldots]$$

or $\qquad A_{xy}=v\text{a}_{xy}-a_{xy}.$

$$\text{Also } A_{\overline{xy}}=A_x+A_y-A_{xy}$$

$$=(1-d\text{a}_x)+(1-d\text{a}_y)-(1-d\text{a}_{xy})$$

$$=1-d(\text{a}_x+\text{a}_y-\text{a}_{xy})$$

$$=1-d\text{a}_{\overline{xy}}.$$

CHAPTER VI

Annual Premiums

1. We have found the value of a whole life assurance of 1 due at the end of the year in which (x) will die, representing the value by the symbol A_x. This is the single premium calculated at a given rate of interest i and on the basis of a given mortality table, which should be paid for the benefit; so that if we have a large homogeneous group of lives each aged exactly x and each paying A_x into a fund which will earn interest at rate i, the fund will be able to pay 1 at the end of the year of death of each member and the last such payment will exhaust the fund if the lives fail in accordance with the given mortality table.

Having deduced a proper A_x for this group of lives, we may find that some members cannot afford to pay the full premium in advance. Each such member might prefer to have his share of the cost spread uniformly over his lifetime.

In the case of the single premium we made each member of the group aged x pay the average cost of the amount assured. We now carry the same principle over to the case of annual premiums and say that the present value of what each member shall pay must be equal to the present value of what he shall receive. Accordingly if P_x represents the annual premium payable by a man now aged x for a whole life assurance of 1 to be paid at the end of the year of his death, we have

$$P_x . \text{a}_x = A_x$$

assuming that the annual premiums are paid in advance each year. This principle that the value of the single premium is equal to the value of all the annual premiums is fundamental.

A_x is the cash value of the benefit to be received by the assured:

$P_x . \text{a}_x$ is the cash value of the annual payments necessary to secure that benefit.

Similarly
$$P_{\overset{1}{x}\,\overline{n}|}\,\mathrm{a}_{x\overline{n}|}=A_{\overset{1}{x}\,\overline{n}|}$$
$$P^{\,1}_{x\overline{n}|}\,\mathrm{a}_{x\overline{n}|}=A^{\,1}_{x\overline{n}|}$$
$$P_{x\overline{n}|}\,\mathrm{a}_{x\overline{n}|}=A_{x\overline{n}|}.$$

The student should note from the above statements the exact meanings of the symbols for the different annual premiums.

2. From $P_x.\mathrm{a}_x=A_x$ and $1=d\mathrm{a}_x+A_x$, we can, by algebra, deduce
$$P_x=\frac{1}{\mathrm{a}_x}-d=\frac{dA_x}{1-A_x}.$$

If a man aged exactly x were to borrow 1 for the period of his life and agree to pay P_x+d in advance each year so long as he lived, the lender would receive interest for each year including the year of death together with the annual payment P_x which is by definition equivalent in value to the return of the capital at the end of the year of death, or $1=(P_x+d)\mathrm{a}_x$.

If a man aged exactly x were to borrow A_x, use it as a single premium for an assurance of 1, agree to pay interest on the loan each year in advance, and charge his policy with the repayment of the loan at the end of the year of his death, he would be paying dA_x each year in advance for a policy of $1-A_x$ or $dA_x=P_x(1-A_x)$.

3. Schedule on opposite page.

4. Any two of these fundamental links
$$1=d\mathrm{a}_x+A_x$$
$$1=(P_x+d)\ (1+a_x)$$
$$dA_x=P_x(1-A_x)$$
$$A_x=P_x.\mathrm{a}_x$$

may be deduced by algebra from the other two. The student should make sure that he can do this.

In some cases two different explanations may be given of any one connecting link, since the product of any two ordinary symbols may be translated into two different statements; for instance,

3. SCHEDULE OF THE CONNECTIONS BETWEEN A_x, a_x, P_x AND d

	future life time of man aged x					year of death	
	x	$x+1$	$x+2$				
a_x = the series	1	1	1	....	1	1		...(1)
A_x = the series	P_x	P_x	P_x	....	P_x	P_x	$=1$	...(2)
1 = the series	d	d	d	....	d	d	$+1$	...(3)
1 = the series	a_x^{-1}	a_x^{-1}	a_x^{-1}	....	a_x^{-1}	a_x^{-1}		...(4) from (1)
A_x = the series	$\mathrm{a}_x^{-1}-d$	$\mathrm{a}_x^{-1}-d$	$\mathrm{a}_x^{-1}-d$	....	$\mathrm{a}_x^{-1}-d$	$\mathrm{a}_x^{-1}-d$	$=1$	...(5) from (3) and (4)
A_x = the series	dA_x	dA_x	dA_x	....	dA_x	dA_x	$+A_x$	...(6) from (3)
The series	P_x-dA_x	P_x-dA_x	P_x-dA_x	....	P_x-dA_x	P_x-dA_x	$=A_x$	...(7) from (2) and (6)
The series	dA_x	dA_x	dA_x	....	dA_x	dA_x	$=1-A_x$	...(8) from (2) and (7)

(1) is a definition of a_x
(2) is a definition of A_x and P_x and shows that $A_x = P_x \cdot \mathrm{a}_x$
(3) is a statement of a loan of 1 for the life of (x)
(4) is a statement that $\mathrm{a}_x \times \mathrm{a}_x^{-1} = 1$
(5) is a statement which when compared with (2) shows that $P_x = \mathrm{a}_x^{-1} - d$
(6) is a statement of a loan of A_x for the life of (x)
(7) is a statement which when compared with (2) shows that $P_x \cdot A_x = P_x - dA_x$
(8) shows that $dA_x = P_x(1 - A_x)$

$P_x . \mathrm{a}_x \equiv$ the value of an annuity-due of P_x per annum on the life of x,

$\equiv$ the annual premium for a whole life assurance of a_x.

or, again

$dA_x \equiv$ the annual interest due in advance each year on a loan of A_x

$\equiv$ the single premium for a whole life assurance of d,

and so on.

The above links may be modified and meanings can generally be given to the modifications.

For example, $A_x = 1 - d\mathrm{a}_x$, which means that if the unit payable under the policy on (x) were payable at once its value would be 1, but as it is not payable until the end of the year of death we must deduct the value of the interest throughout the lifetime of (x).

Also, for example, $A_x = v(1+a_x) - a_x$, which means that if (x) was entitled to a payment of 1 at the end of each year he began which is equivalent to v at the beginning of each such year, and (x) was obliged to pay 1 at the end of each year he completed, his property in regard to these annuities would be only the value of that 1 due at the end of the year which he will ultimately begin but not complete.

5. For a term shorter than the whole of life, we have

$$1 = (P_{x\overline{n|}} + d)\mathrm{a}_{x\overline{n|}}$$

but note $1 = (P^{1}_{x\overline{n|}} + d)\mathrm{a}_{x\overline{n|}} + A_{x\overline{n|}}^{\;\;1}$

and $1 = (P_{x\overline{n|}}^{\;\;1} + d)\mathrm{a}_{x\overline{n|}} + A^{1}_{x\overline{n|}}$.

Also $dA_{x\overline{n|}} = P_{x\overline{n|}}(1 - A_{x\overline{n|}})$

but note $dA^{1}_{x\overline{n|}} + P_{x\overline{n|}}^{\;\;1} . A^{1}_{x\overline{n|}} = P^{1}_{x\overline{n|}}(1 - A^{1}_{x\overline{n|}})$

and $dA_{x\overline{n|}}^{\;\;1} + P^{1}_{x\overline{n|}} . A_{x\overline{n|}}^{\;\;1} = P_{x\overline{n|}}^{\;\;1}(1 - A_{x\overline{n|}}^{\;\;1})$.

6. It will be noticed, in the case of whole life and endowment assurances, that there is a very simple relation between the annual premium for the assurance and the value of the corresponding annuity.

For example $P_x = \dfrac{1}{1+a_x} - d$

and $P_{x\overline{n}|} = \dfrac{1}{1+a_{x\,\overline{n-1}|}} - d.$

Hence if α represents the annual premium and β the value of the corresponding annuity, we have

$$\alpha = \frac{1}{1+\beta} - d.$$

Since the range of β is limited to values lying between 0 and $\frac{1}{i}$ we can prepare tables, one for each rate of interest desired, showing the value of α corresponding to values of β beginning at 0 and rising, by differences of .01 for example, up to $\frac{1}{i}$. Such a table is called an Annual Premium Conversion Table.

If we are given an annual premium conversion table at rate i and the value of an annuity at rate j, say $a_{(j)}$, and it is required to find the corresponding annual premium at rate j, say $P_{(j)}$, we have on entering the given conversion table with $a_{(j)}$ a result

$$\alpha = \frac{1}{1+a_{(j)}} - d_{(i)}$$

but

$$P_{(j)} = \frac{1}{1+a_{(j)}} - d_{(j)}$$

$$\therefore P_{(j)} = \alpha + (d_{(i)} - d_{(j)}).$$

7. The annual premium limited to n payments for a whole life policy for 1 is written ${}_nP_x$.

Therefore $A_x = {}_nP_x \cdot \text{a}_{x\overline{n}|}$ and hence ${}_nP_x = \dfrac{dA_x}{1-A_{x\overline{n}|}}$.

Similarly $A_{x\,\overline{n}|} = {}_mP_{x\,\overline{n}|}\,.\,\text{a}_{x\,\overline{m}|}$ and hence ${}_mP_{x\,\overline{n}|} = \dfrac{dA_{x\,\overline{n}|}}{1-A_{x\,\overline{m}|}}$,

$$A^{1}_{x\,\overline{n}|} = {}_mP^{1}_{x\,\overline{n}|}\,.\,\text{a}_{x\,\overline{m}|} \quad \text{and hence} \quad {}_mP^{1}_{x\,\overline{n}|} = \frac{dA^{1}_{x\,\overline{n}|}}{1-A_{x\,\overline{m}|}},$$

and

$$A_{x\,\overset{1}{\overline{n}|}} = {}_mP_{x\,\overset{1}{\overline{n}|}}\,.\,\text{a}_{x\,\overline{m}|} \quad \text{and hence} \quad {}_mP_{x\,\overset{1}{\overline{n}|}} = \frac{dA_{x\,\overset{1}{\overline{n}|}}}{1-A_{x\,\overline{m}|}}.$$

8. The premium may be payable half-yearly or quarterly, and the benefit may be payable at the end of the month or the quarter in which the life may fail, instead of at the end of the year of death.

P is always an annual premium payable in advance, but $P^{(m)}_x$ indicates that an mth of this premium is payable at the beginning of each mth part of a year, so long as (x) may live, and ${}^{(n)}P^{(m)}_x$ indicates further that the benefit will be paid at the end of that nth part of a year in which (x) may die.

Thus $$P^{(m)}_x\text{a}^{(m)}_x = P^{(m)}_x\left(\frac{1}{m}+a^{(m)}_x\right) = A_x,$$

so that $$P^{(m)}_x = \frac{A_x}{\text{a}^{(m)}_x} = \frac{A_x}{\text{a}_x - \dfrac{m-1}{2m}} = \frac{P_x}{1-\dfrac{m-1}{2m}\cdot\dfrac{1}{\text{a}_x}} = \frac{P_x}{1-\dfrac{m-1}{2m}(P_x+d)}.$$

Whence $$P^{(m)}_x\left[1-\frac{m-1}{2m}(P_x+d)\right] = P_x$$

or $$P^{(m)}_x = P_x + \frac{m-1}{2m}P^{(m)}_x(P_x+d).$$

Thus $P^{(m)}_x$ exceeds P_x by the sum of the two terms

$$\left[\frac{m-1}{2m}P^{(m)}_x\right]P_x \quad \text{and} \quad \left[\frac{m-1}{2m}P^{(m)}_x\right]d$$

the meanings of which should be noted.

Also $$P^{(m)}_{x\,\overline{n}|} \cdot \text{a}^{(m)}_{x\,\overline{n}|} = P^{(m)}_{x\,\overline{n}|}\left(\frac{1}{m} + a^{(m)}_{x:\overline{n-\frac{1}{m}}|}\right) = A_{x\,\overline{n}|}$$

$${}^{(r)}P^{(m)}_x \cdot \text{a}^{(m)}_x = A^{(r)}_x$$

$${}^{(r)}_{n}P^{(m)}_{x\,\overline{t}|} \cdot \text{a}^{(m)}_{x\,\overline{n}|} = A^{(r)}_{x\,\overline{t}|}$$

$$\overline{P}_x\,\bar{a}_x = \overline{A}_x \text{ and } {}^{\infty}\overline{P}_x\,\bar{a}_x = \overline{A}_x$$

9. Again, the annual premium may be payable by instalments and the full number of instalments become payable for each year on which (x) enters. If such a premium is payable by m instalments it is written $P^{[m]}_x$,

so that $P^{[m]}_x . \text{a}^{(m)}_{\overline{1}|} = P_x$ or $P^{[m]}_x \text{a}^{(m)}_x + \dfrac{m-1}{2m}\, P^{[m]}_x A_x = A_x$

$${}^{(n)}P^{[m]}_x \,.\, \text{a}^{(m)}_{\overline{1}|} = {}^{(n)}P_x$$

and ${}^{(\infty)}P^{[m]}_x .\; \text{a}^{(m)}_{\overline{1}|} = {}^{(\infty)}P_x$

where $\text{a}^{(m)}_{\overline{1}|}$ may be obtained from the relation $f_{(m)}\text{a}^{(m)}_{\overline{1}|} = d$.

When the annual premium is thus paid by instalments a deduction is made from the sum assured equivalent to the value of the unpaid instalments due in the year of death.

10. Since $1 = (P_x + d)\text{a}_x = (P_{x+1} + d)\text{a}_{x+1}$

and $a_x = \text{a}_x - 1 = vp_x\text{a}_{x+1}$

we have $P_{x+1} = P_x + (P_x - vq_x)\dfrac{1}{a_x} = P_x + (P_{x+1} - vq_x)\dfrac{1}{\text{a}_x}$

or $P_x\,(1+i) = q_x + p_x(P_{x+1} - P_x)\text{a}_{x+1}$.

Each of these has a meaning that can be expressed in words.

11. Consider two lives (x) and (y).

For an assurance payable on the first death, that is when the pair is broken, we have

$$P_{xy}\text{a}_{xy} = A_{xy}.$$

For an assurance payable on the second death, that is on the death of the survivor, we have

$$P_{\overline{xy}}\mathrm{a}_{\overline{xy}} = A_{\overline{xy}}.$$

But the annual premium payable only during the joint lives for the benefit indicated by $A_{\overline{xy}}$ would be $\dfrac{A_{\overline{xy}}}{\mathrm{a}_{xy}}$.

CHAPTER VII

Commutation Symbols

1. Up to this point we have been developing a theory in terms of fundamental symbols without making any attempt to ascertain arithmetical values. If we are to apply our theories we must be able to compute the numerical values of the symbols we have been using.

2. Let us attempt to find the numerical value of A_{25} which is the value now of an assurance of 1 payable at the end of the year of the death of a life now aged exactly 25. We have

$$A_{25}=\frac{1}{l_{25}}[vd_{25}+v^2d_{26}+v^3d_{27}+\ldots].$$

The values of d and the values of v are tabulated and we can perform the series of multiplications, add our results and divide by the value of l_{25}. This process is a tedious one, but it can be done. Before doing it we should do well to look ahead a little. We shall next wish to find the value A_{26} and we have

$$A_{26}=\frac{1}{l_{26}}[vd_{26}+v^2d_{27}+v^3d_{28}+\ldots].$$

Not one of our former multiplications, and there were many of them, is going to be of use again.

If we do obtain one value of A_x we might use the fact that $(1+i)\,A_x=q_x+p_xA_{x+1}$ to obtain, one by one, the values at younger or older ages.

But we can do better, for in general

$$A_x=\frac{1}{l_x}[vd_x+v^2d_{x+1}+v^3d_{x+2}+\ldots]$$

$$=\frac{1}{v^xl_x}[v^{x+1}d_x+v^{x+2}d_{x+1}+v^{x+3}d_{x+2}+\ldots].$$

Writing C_x for $v^{x+1}d_x$, D_x for $v^x l_x$

and M_x for $C_x+C_{x+1}+C_{x+2}+\ldots$ to end of table,

we have $A_x=\dfrac{M_x}{D_x}$, for all values of x. The whole series of values of A_x for any mortality table at any rate of interest can be found at once after the columns M_x and D_x have been formed for that table and at that rate of interest.

3. Similarly for annuity values

$$a_x=\frac{1}{l_x}[vl_{x+1}+v^2l_{x+2}+v^3l_{x+3}+\ldots]$$

$$=\frac{1}{v^x l_x}[v^{x+1}l_{x+1}+v^{x+2}l_{x+2}+v^{x+3}l_{x+3}+\ldots]$$

$$=\frac{1}{D_x}[D_{x+1}+D_{x+2}+D_{x+3}+\ldots]$$

and writing N_x for $D_{x+1}+D_{x+2}+D_{x+3}+\ldots$

or $\mathbf{N}_x$ for $D_x+D_{x+1}+D_{x+2}+\ldots$

we have $a_x=\dfrac{N_x}{D_x}$, and $\mathrm{a}_x=\dfrac{\mathbf{N}_x}{D_x}$.

The difference between N_x and $\mathbf{N}_x$ must be noted

$$\mathbf{N}_x=D_x+N_x=N_{x-1}.$$

Also since $d_x=l_x-l_{x+1}$

$$v^{x+1}d_x=v^{x+1}l_x-v^{x+1}l_{x+1}$$

therefore $C_x=vD_x-D_{x+1}$

and $M_x=vN_{x-1}-N_x$

$$=v\mathbf{N}_x-\mathbf{N}_{x+1}.$$

4. The following numerical illustration of the deduction of values for a_x from known values of l_x and v^x may be helpful.

ANNUITY VALUES

English Life Table No. 8—Males—3%

x	l_x	v^x	$l_x \times v^x = D_x$	$\sum_{n=1}^{\infty} D_{x+n} = N_x$	$N_x \div D_x = a_x$
(1)	(2)	(3)	(4)	(5)	(6)
1	879,559.	.970874	853,941.	21,439,847.	25.107
2	849,444.	.942596	800,682.	20,639,165.	25.777
3	838,091.	.915142	766,972.	19,872,193.	25.910
4	831,235.	.888487	738,541.	19,133,652.	25.907
..	......	......	....	....	......
..	......	......	....	....	......
..	......	......	....	....	......
..	......	......	....	....	......
71	312,679.	.122619	38,340.	243,284.	6.345
72	290,752.	.119047	34,613.	208,671.	6.029
73	268,618.	.115580	31,047.	177,624.	5.721
74	246,395.	.112214	27,649.	149,975.	5.424
..	......	......	....	....	......
..	......	......	....	....	......

Columns (2) and (3) are known and available values.

Column (4) is the result of (2)×(3) for each value of x.

Column (5) is made from the bottom upwards. Each value in it is the sum of all the succeeding values in column (4).

Column (6) is the result of (5)÷(4) for each value of x.

5. Similarly the values of A_x can be found as indicated in the following schedule:

x	d_x	v^{x+1}	$(2)\times(3) = C_x$	$\sum_{n=0}^{\infty} C_{x+n} = M_x$	$(5) \div D_x = A_x$
(1)	(2)	(3)	(4)	(5)	(6)
..	..	..	..	..	..
..	..	..	..	..	..

6. We now have $D_x = v^x l_x$

$$C_x = v^{x+1} d_x = v^{x+1}(l_x - l_{x+1}) = vD_x - D_{x+1}$$

$$\mathbf{N}_{x+1} = N_x = D_{x+1} + D_{x+2} + D_{x+3} + \ldots$$

$$M_x = C_x + C_{x+1} + C_{x+2} + \ldots$$

$$= v\mathbf{N}_x - \mathbf{N}_{x+1} = vN_{x-1} - N_x.$$

If we write $$R_x = M_x + M_{x+1} + M_{x+2} + \ldots$$

$$= C_x + 2C_{x+1} + 3C_{x+2} + \ldots$$

and either $$S_x = N_x + N_{x+1} + N_{x+2} + \ldots$$

$$= D_{x+1} + 2D_{x+2} + 3D_{x+3} + \ldots$$

or $$\mathbf{S}_x = \mathbf{N}_x + \mathbf{N}_{x+1} + \mathbf{N}_{x+2} + \ldots$$

$$= D_x + 2D_{x+1} + 3D_{x+2} + \ldots$$

we have $$R_x = v\mathbf{S}_x - \mathbf{S}_{x+1} = vS_{x-1} - S_x.$$

7. The arithmetical values of these commutation symbols are tabulated, but in using such values the student should be careful to see whether the sum of the D_x values is put down as an N-value or as an $\mathbf{N}$-value. If the N-value is used and these values are summed the column of sums will be an S-value. If the $\mathbf{N}$-value is used it will be followed by an $\mathbf{S}$-value.

8. We have seen that $A_x = \dfrac{M_x}{D_x}$

and hence $${}_{n|}A_x = \frac{1}{D_x}\left[C_{x+n} + C_{x+n+1} + C_{x+n+2} + \ldots\right]$$

$$= \frac{M_{x+n}}{D_x} = \frac{D_{x+n}}{D_x} A_{x+n}.$$

Similarly $$A^{1}_{x\,\overline{n}|} = {}_{|n}A_x = \frac{1}{D_x}\left[C_x + C_{x+1} + \ldots + C_{x+n-1}\right]$$

$$= \frac{M_x - M_{x+n}}{D_x}$$

but $$A_{x\,\overset{1}{\overline{n|}}} = v^n{}_np_x = \frac{D_{x+n}}{D_x}.$$

Therefore $$A_{x\overline{n|}} = A^1_{x\overline{n|}} + A_{x\,\overset{1}{\overline{n|}}} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}.$$

9. Also,

$$_{n|}a_x = \frac{N_{x+n}}{D_x} = \frac{\mathbf{N}_{x+n+1}}{D_x} = \frac{D_{x+n}}{D_x}.a_{x+n}$$

$$a_{x\overline{n|}} = \frac{N_x - N_{x+n}}{D_x} = \frac{\mathbf{N}_{x+1} - \mathbf{N}_{x+n+1}}{D_x}$$

$$_{n|}\mathrm{a}_x = \frac{\mathbf{N}_{x+n}}{D_x} = \frac{N_{x+n-1}}{D_x} = \frac{D_{x+n}}{D_x}\mathrm{a}_{x+n}$$

and hence $$\mathrm{a}_{x\overline{n|}} = \frac{\mathbf{N}_x - \mathbf{N}_{x+n}}{D_x} = \frac{N_{x-1} - N_{x+n-1}}{D_x}.$$

10. For annual premiums, we have,

$P_x \cdot \mathrm{a}_x = A_x$ so that $P_x \cdot \mathbf{N}_x = M_x$

$P^1_{x\overline{n|}} \cdot \mathrm{a}_{x\overline{n|}} = A^1_{x\overline{n|}}$ so that $P^1_{x\overline{n|}}(\mathbf{N}_x - \mathbf{N}_{x+n}) = M_x - M_{x+n}$

$P_{x\,\overset{1}{\overline{n|}}} \cdot \mathrm{a}_{x\overline{n|}} = A_{x\,\overset{1}{\overline{n|}}}$ so that $P_{x\,\overset{1}{\overline{n|}}}(\mathbf{N}_x - \mathbf{N}_{x+n}) = D_{x+n}$

$P_{x\overline{n|}} \cdot \mathrm{a}_{x\overline{n|}} = A_{x\overline{n|}}$ so that $P_{x\overline{n|}}(\mathbf{N}_x - \mathbf{N}_{x+n}) = M_x - M_{x+n} + D_{x+n}$

and ${}_nP_x \cdot \mathrm{a}_{x\overline{n|}} = A_x$ so that ${}_nP_x(\mathbf{N}_x - \mathbf{N}_{x+n}) = M_x$.

11. The commutation symbols also enable us to obtain workable expressions for increasing or decreasing assurances and increasing or decreasing annuities.

$$\begin{aligned}(IA)_x &= \frac{1}{l_x}[vd_x + 2v^2d_{x+1} + 3v^3d_{x+2} + 4v^4d_{x+3} + \ldots] \\ &= \frac{1}{D_x}[C_x + 2C_{x+1} + 3C_{x+2} + 4C_{x+3} + \ldots] \\ &= \frac{1}{D_x}[M_x + M_{x+1} + M_{x+2} + \ldots] \\ &= \frac{R_x}{D_x}.\end{aligned}$$

So $$(IA)^{1}_{x\overline{n}|} = \frac{1}{D_x}[C_x+2C_{x+1}+3C_{x+2}+\ldots+nC_{x+n-1}]$$

$$= \frac{1}{D_x}\left[(M_x-M_{x+n})+(M_{x+1}-M_{x+n})+\ldots+(M_{x+n-1}-M_{x+n})\right]$$

$$= \frac{1}{D_x}[R_x-R_{x+n}-nM_{x+n}].$$

Also $$(Ia)_x = \frac{1}{l_x}[vl_{x+1}+2v^2l_{x+2}+3v^3l_{x+3}+\ldots]$$

$$= \frac{1}{D_x}[D_{x+1}+2D_{x+2}+3D_{x+3}+\ldots]$$

$$= \frac{1}{D_x}[N_x+N_{x+1}+N_{x+2}+\ldots]$$

$$= \frac{S_x}{D_x}.$$

So $$(Ia)_{x\overline{n}|} = \frac{1}{D_x}[D_{x+1}+2D_{x+2}+3D_{x+3}+\ldots+nD_{x+n}]$$

$$= \frac{1}{D_x}\left[(N_x-N_{x+n})+(N_{x+1}-N_{x+n})+\ldots+(N_{x+n-1}-N_{x+n})\right]$$

$$= \frac{1}{D_x}[S_x-S_{x+n}-nN_{x+n}].$$

Similarly

$$(I\mathrm{a})_x = \frac{\mathrm{S}_x}{D_x}$$

and $$(I\mathrm{a})_{x\overline{n}|} = \frac{1}{D_x}(\mathrm{S}_x-\mathrm{S}_{x+n}-n\mathrm{N}_{x+n}).$$

12. Suitable commutation symbols may be made for the valuation of assurance and annuity functions involving joint lives. For instance

$$a_{xy} = \frac{1}{l_{xy}}[vl_{x+1:y+1}+v^2l_{x+2:y+2}+\ldots].$$

Assuming that $x>y$, we define

$$D_{xy}=v^x l_x l_y$$

and $$N_{xy}=D_{x+1:y+1}+D_{x+2:y+2}+\ldots$$

so that $$a_{xy}=\frac{N_{xy}}{D_{xy}}.$$

Also $$A_{xy}=\frac{1}{l_{xy}}[vd_{xy}+v^2d_{x+1:y+1}+\ldots].$$

Assuming that $x>y$, we define

$$C_{xy}=v^{x+1}d_{xy}=v^{x+1}(l_{xy}-l_{x+1:y+1})$$

and $$M_{xy}=C_{xy}+C_{x+1:y+1}+\ldots$$

so that $$A_{xy}=\frac{M_{xy}}{D_{xy}}$$

The student should try the result of making $D_{xy}=v^{\frac{x+y}{2}}.l_x.l_y$.

13. We have $\bar{A}_x=\int_0^\infty v^t{}_tp_x\mu_{x+t}dt$

$$=\frac{1}{l_x}\int_0^\infty v^t l_{x+t}\mu_{x+t}dt$$

$$=\frac{1}{D_x}\int_0^\infty D_{x+t}\mu_{x+t}dt.$$

We write $\bar{C}_x=\int_0^1 D_{x+t}\mu_{x+t}dt$

and $$\bar{M}_x=\bar{C}_x+\bar{C}_{x+1}+\bar{C}_{x+2}+\ldots=\int_0^\infty D_{x+t}\mu_{x+t}dt$$

so that $$\bar{A}_x=\frac{\bar{M}_x}{D_x}.$$

For practical work, since, $\bar{C}_x=\int_0^1 D_{x+t}\mu_{x+t}dt$,

we take $$\bar{C}_x=1\times D_{x+\frac{1}{2}}\mu_{x+\frac{1}{2}}, \text{ approximately}$$

$$=v^{x+\frac{1}{2}}l_{x+\frac{1}{2}}\left(\frac{-1}{l_{x+\frac{1}{2}}}\frac{dl_{x+\frac{1}{2}}}{dx}\right)$$

$$=v^{x+\frac{1}{2}}\left(\frac{l_x-l_{x+1}}{1}\right)\text{ approximately}$$

$$=v^{x+\frac{1}{2}}d_x$$

$$=(1+i)^{\frac{1}{2}}C_x$$

or $$\overline{C}_x=(1+\tfrac{1}{2}i)C_x,\text{ approximately,}$$

and $$\overline{M}_x=(1+\tfrac{1}{2}i)M_x\text{ approximately}$$

so that $$\overline{A}_x=\frac{\overline{M}_x}{D_x}=\frac{(1+\frac{1}{2}i)M_x}{D_x}=(1+\tfrac{1}{2}i)A_x.$$

14. We have $\bar{a}_x=\int_0^\infty v^t{}_tp_x\,.\,dt=\frac{1}{l_x}\int_0^\infty v^t l_{x+t}\,.\,dt=\frac{1}{D_x}\int_0^\infty D_{x+t}\,.\,dt.$

Now, if we call $\int_0^1 v^{x+t}l_{x+t}\,.\,dt=\overline{D}_x$

and put $\overline{N}_x$ for $\overline{D}_x+\overline{D}_{x+1}+\overline{D}_{x+2}+\ldots$

it follows that $\bar{a}_x=\dfrac{\overline{N}_x}{D_x}.$

Since $\overline{D}_x=\int_0^1 D_{x+t}\,.\,dt,$

$\overline{D}_x$ is given in practical work its approximate value

$$D_{x+\frac{1}{2}}=\tfrac{1}{2}(D_x+D_{x+1}).$$

So that $$\overline{N}_x=\tfrac{1}{2}(D_x+D_{x+1})+\tfrac{1}{2}(D_{x+1}+D_{x+2})+\ldots$$

$$=\tfrac{1}{2}D_x+D_{x+1}+D_{x+2}+\ldots$$

$$=\tfrac{1}{2}D_x+N_x$$

$$=\mathrm{N}_x-\tfrac{1}{2}D_x$$

and $$\bar{a}_x=\frac{\frac{1}{2}D_x+N_x}{D_x}=\tfrac{1}{2}+a_x,\text{ as before.}$$

CHAPTER VIII

Policy Values

1. Consider l_x persons each aged exactly x and assume that each person takes out a whole life assurance for 1 at a single premium of A_x. An initial fund of l_xA_x is set up. The interest earned by the end of the year is il_xA_x and the total accumulated amount of the fund at the end of the year is $(1+i)l_xA_x$.

Out of this sum there are d_x claims of 1 each to be paid. Therefore, the l_{x+1} survivors will have to their credit

$$\begin{aligned}(1+i)l_xA_x-d_x &= (1+i)\ (vd_x+v^2d_{x+1}+\ \dots\ )-d_x\\ &= vd_{x+1}+v^2d_{x+2}+\ \dots\\ &= l_{x+1}A_{x+1}\end{aligned}$$

and the share of each will be A_{x+1}. This is exactly the premium which a new man of the same age $x+1$ must pay if he wishes to join the fund a year after it started.

Or again, consider (x) who pays a single premium A_x for a whole life assurance of 1.

If (x) should die during the year, the probability of which is q_x, a sum of 1 will become payable. If (x) should survive the year, the probability of which is p_x, there must be A_{x+1} to his credit in the fund,

therefore $$A_x=vq_x\times 1+vp_xA_{x+1}$$

or $$A_x(1+i)=A_{x+1}+q_x(1-A_{x+1}).$$

This statement has already been proved. It means that at the end of the year the fund must have on hand in respect of this policy A_{x+1} and further, in the case of death, it will have to provide the difference between the claim of 1 and the amount held A_{x+1} or a net sum of $(1-A_{x+1})$.

The amount, $1-A_{x+1}$, which is the difference between the sum assured and the value of the policy at age $x+1$, is called the "net amount at risk" for the year x to $x+1$,

$q_x(1-A_{x+1})$ is called the "cost of insurance" or "death strain" for the year x to $x+1$.

2. Consider l_x persons each aged exactly x and assume that each person takes out a whole life assurance for 1 at an annual premium of P_x.

The fund starts with l_xP_x which initial sum accumulated at interest rate i amounts to $l_xP_x(1+i)$ at the end of the first year. The claims for the first year amount to $1\times d_x=d_x$, so that the net amount available for l_{x+1} survivors at the end of the first year is $l_xP_x(1+i)-d_x$.

At the beginning of the second year, premiums aggregating $l_{x+1}P_x$ are paid into the fund. The initial amount of the fund for the second year then stands at

$$l_xP_x(1+i)-d_x+l_{x+1}P_x$$

which at interest amounts at the end of the year to

$$\left[l_xP_x(1+i)-d_x+l_{x+1}P_x\right](1+i).$$

Deducting claims for the d_{x+1} deaths in that year, we have the net amount available for the l_{x+2} survivors equal to

$$\left[l_xP_x(1+i)-d_x+l_{x+1}P_x\right](1+i)-d_{x+1}.$$

If this process be continued for n years, there will be available at the end of the n years for the l_{x+n} survivors a total of

$$\begin{aligned}
&l_xP_x(1+i)^n-d_x(1+i)^{n-1}+l_{x+1}P_x(1+i)^{n-1}-d_{x+1}(1+i)^{n-2}+\\
&\qquad\qquad \ldots +l_{x+n-1}P_x(1+i)-d_{x+n-1}\\
&=P_x\left[l_x(1+i)^n+l_{x+1}(1+i)^{n-1}+\ldots\ldots+l_{x+n-1}(1+i)\right]\\
&\qquad -\left[d_x(1+i)^{n-1}+d_{x+1}(1+i)^{n-2}+\ldots+d_{x+n-1}\right]\\
&=P_x(1+i)^{x+n}[D_x+D_{x+1}+\ldots+D_{x+n-1}]\\
&\qquad -(1+i)^{x+n}[C_x+C_{x+1}+\ldots+C_{x+n-1}]\\
&=P_x(1+i)^{x+n}(\mathrm{N}_x-\mathrm{N}_{x+n})-(1+i)^{x+n}(M_x-M_{x+n})
\end{aligned}$$

or an average amount per person of

$$\frac{1}{l_{x+n}}\left[P_x(1+i)^{x+n}(\mathrm{N}_x-\mathrm{N}_{x+n})-(1+i)^{x+n}(M_x-M_{x+n})\right]$$

$=P_x\left(\frac{N_x-N_{x+n}}{D_{x+n}}\right)-\left(\frac{M_x-M_{x+n}}{D_{x+n}}\right)$. This is called the "reserve" at the end of n years and is written ${}_nV_x$.

But $P_x.N_x=M_x$.

Therefore $\quad {}_nV_x=\dfrac{M_{x+n}-P_x\cdot N_{x+n}}{D_{x+n}}$

$=A_{x+n}-P_x.a_{x+n}$, which is the value of the benefit ultimately payable less the value of the premiums still to be paid.

The value or reserve at the end of n years on a whole life policy with annual premiums, can be obtained from two different points of view.

From the retrospective point of view, the reserve is the difference between the premiums paid in and the claims paid out, each accumulated at interest, with due allowance made for mortality.

From the prospective point of view, the reserve is the difference between the value of the future benefit and the value of the future premiums.

These two methods will produce the same result provided the same bases of interest and mortality are used in the valuation factors as were used in the deduction of the annual premium.

3. Again, consider a whole life policy taken out on (x) with an annual premium of P_x.

The present value at age x of the n premiums which may be paid between the ages x and $x+n$ is $P_x.a_{x\,\overline{n}|}$.

The present value at age x of the benefit obtainable during these n years following age x is $A^{1}_{x\,\overline{n}|}$.

Then the difference $P_x.a_{x\,\overline{n}|}-A^{1}_{x\,\overline{n}|}$ represents the value at age x of the excess amount that may be paid in over and above the cost of the assurance for that n year period. But the value at age x of the reserve that may be held at age $x+n$ is $v^n{}_np_x.{}_nV_x$

therefore ${}_nV_x=(1+i)^n(P_x.a_{x\,\overline{n}|}-A^{1}_{x\,\overline{n}|})\,{}_np_x^{-1}$

$$=\frac{D_x}{D_{x+n}}(P_x.a_{x\,\overline{n}|}-A^{1}_{x\,\overline{n}|})$$

$$=\frac{D_x}{D_{x+n}}\left[P_x\left(\frac{\mathrm{N}_x-\mathrm{N}_{x+n}}{D_x}\right)-\left(\frac{M_x-M_{x+n}}{D_x}\right)\right]$$

$$=\frac{P_x(\mathrm{N}_x-\mathrm{N}_{x+n})-(M_x-M_{x+n})}{D_{x+n}}\text{, as before.}$$

4. From the fundamental relations connecting A, a and P and the rate of interest the student should prove that

$$_nV_x=1-(P_x+d)\mathrm{a}_{x+n}$$

$$=1-\frac{\mathrm{a}_{x+n}}{\mathrm{a}_x}=\frac{\mathrm{a}_x-\mathrm{a}_{x+n}}{\mathrm{a}_x}=(P_x+d)\ (\mathrm{a}_x-\mathrm{a}_{x+n})$$

$$=(P_{x+n}-P_x)\mathrm{a}_{x+n}=\frac{P_{x+n}-P_x}{P_{x+n}+d}$$

$$=\frac{A_{x+n}-A_x}{1-A_x}=\frac{A_{x+n}-A_x}{d\mathrm{a}_x}$$

$$=A_{x+n}\left(1-\frac{P_x}{P_{x+n}}\right).$$

Explanations in words should be made for each of the above expressions for $_nV_x$.

5. Again, consider a whole life policy taken out by (x) with an annual premium of P_x

$_nV_x$ is the "terminal reserve" for the nth year

$(_{n-1}V_x+P_x)$ is the "initial reserve" for the nth year

$\frac{1}{2}(_{n-1}V_x+{}_nV_x+P_x)$ is the "mid-year" or "mean reserve" for the nth year

$(1-{}_nV_x)$ is the "net amount at risk" for the nth year and $q_{x+n-1}(1-{}_nV_x)$ is the "cost of insurance" or "death strain" for the nth year.

We have then, in respect of the nth year

$$(_{n-1}V_x+P_x)\ (1+i)-q_{x+n-1}(1-{}_nV_x)={}_nV_x.$$

At the beginning of the nth year, just before the premium is paid, there is $_{n-1}V_x$ to the credit of the policy holder. After this man has paid his premium there will be $_{n-1}V_x+P_x$ to his credit.

This will grow during the year to $({}_{n-1}V_x+P_x)\ (1+i)$. From this amount $q_x(1-{}_nV_x)$ representing the cost of insurance for the year will be deducted leaving ${}_nV_x$ to the credit of the policyholder whether he has survived or not. If the policy becomes a claim an additional credit of $(1-{}_nV_x)$ will be added to the reserve ${}_nV_x$ in order to bring the account up to the amount of the claim.

Algebraically

$$\begin{aligned}
&({}_{n-1}V_x+P_x)\ (1+i)-q_{x+n-1}(1-{}_nV_x)\\
&=\left(1-\frac{\text{a}_{x+n-1}}{\text{a}_x}+\frac{1}{\text{a}_x}-d\right)(1+i)-q_{x+n-1}\left(\frac{\text{a}_{x+n}}{\text{a}_x}\right)\\
&=1-\frac{1}{\text{a}_x}\cdot\frac{\text{a}_{x+n-1}-1}{v}-(1-p_{x+n-1})\frac{\text{a}_{x+n}}{\text{a}_x}\\
&=1-\frac{1}{\text{a}_x}\left[p_{x+n-1}\cdot\text{a}_{x+n}+(1-p_{x+n-1})\,\text{a}_{x+n}\right]\\
&=1-\frac{\text{a}_{x+n}}{\text{a}_x}\\
&={}_nV_x.
\end{aligned}$$

6. Further, rearranging the above identity, we have

$$ {}_nV_x(1-q_{x+n-1})=({}_{n-1}V_x+P_x)\ (1+i)-q_{x+n-1}$$

or $$ {}_nV_x\cdot l_{x+n}=({}_{n-1}V_x+P_x)(1+i)\,l_{x+n-1}-d_{x+n-1}$$

or $$ {}_nV_x=({}_{n-1}V_x+P_x)\ (1+i)p_{x+n-1}^{-1}-(p_{x+n-1}^{-1}-1)$$

or $$ {}_nV_x=({}_{n-1}V_x+P_x)\ u_{x+n-1}-k_{x+n-1}\ \text{(Fackler's formula)}$$

where

$$u_{x+n-1}=(1+i)\,p_{x+n-1}^{-1}=\frac{D_{x+n-1}}{D_{x+n}}\text{and } k_{x+n-1}=(p_{x+n-1}^{-1}-1)=\frac{C_{x+n-1}}{D_{x+n}}.$$

The functions, u and k can be calculated for any given bases of interest and mortality.

Starting with $n=1$, we have ${}_1V_x=P_x\cdot u_x-k_x$.

If $n=2$, we have, ${}_2V_x=({}_1V_x+P_x)u_{x+1}-k_{x+1}$ and so on.

This is a continuous method which will give a complete table of ${}_nV_x$ for each particular value of x. This table can be checked periodically by an independent relation such as ${}_nV_x=1-\dfrac{\text{a}_{x+n}}{\text{a}_x}$.

7. By a similar line of reasoning, analogous values and relations can be deduced for any type of policy.

For an endowment assurance of 1 with annual premiums, we have

$$
\begin{aligned}
{}_tV_{x\overline{n}|} &= A_{x+t:\overline{n-t}|} - P_{x\overline{n}|}\cdot \mathrm{a}_{x+t:\overline{n-t}|} \\
&= \frac{1}{D_{x+t}}\Big[P_{x\overline{n}|}(\mathrm{N}_x - \mathrm{N}_{x+t}) - (M_x - M_{x+t})\Big]
\end{aligned}
$$

and $\qquad ({}_{t-1}V_{x\overline{n}|} + P_{x\overline{n}|})(1+i) - q_{x+t-1}(1 - {}_tV_{x\overline{n}|}) = {}_tV_{x\overline{n}|}.$

For an n year term assurance of 1 with annual premiums, we have

$$
\begin{aligned}
{}_tV^1_{x\overline{n}|} &= A^{\;1}_{x+t:\overline{n-t}|} - P^1_{x\overline{n}|}\cdot \mathrm{a}_{x+t:\overline{n-t}|} \\
&= \frac{1}{D_{x+t}}\Big[P^1_{x\overline{n}|}(\mathrm{N}_x - \mathrm{N}_{x+t}) - (M_x - M_{x+t})\Big]
\end{aligned}
$$

and $\qquad ({}_{t-1}V^1_{x\overline{n}|} + P^1_{x\overline{n}|})\ (1+i) - q_{x+t-1}(1 - {}_tV^1_{x\overline{n}|}) = {}_tV^1_{x\overline{n}|}.$

For an n-year limited payment whole life assurance of 1 we have, when $t<n$

$$
\begin{aligned}
{}_{t:n}V_x &= A_{x+t} - {}_nP_x\cdot \mathrm{a}_{x+t:\overline{n-t}|} \\
&= \frac{1}{D_{x+t}}\Big[{}_nP_x\ (\mathrm{N}_x - \mathrm{N}_{x+t}) - (M_x - M_{x+t})\Big]
\end{aligned}
$$

and $\qquad ({}_{t-1:n}V_x + {}_nP_x)\ (1+i) - q_{x+t-1}(1 - {}_{t:n}V_x) = {}_{t:n}V_x.$

Also, when $t \geqq n$

$$
\begin{aligned}
{}_{t:n}V_x &= A_{x+t} \\
&= \frac{1}{D_{x+t}}\Big[{}_nP_x(\mathrm{N}_x - \mathrm{N}_{x+n}) - (M_x - M_{x+t})\Big].
\end{aligned}
$$

Also if $t>n$, $A_{x+t-1}(1+i) - q_{x+t-1}(1 - A_{x+t}) = A_{x+t}.$

But when $t=n$, we have

$$({}_{n-1:n}V_x + {}_nP_x)\ (1+i) - q_{x+n-1}(1 - A_{x+n}) = A_{x+n}.$$

Expressions for the reserve under endowment, term and limited payment policies in terms of the A, a and P symbols analogous to those in section 4 should be obtained by the student.

8. Policy values can be written down for policies with premiums payable more often than once a year and/or policies where the sum assured is payable at the end of that mth part of the year in which the death occurs.

Thus ${}_nV_x^{(m)}$ represents the reserve after n years on a whole life policy for 1 due at the end of the year of death where the premiums are payable m times a year, so that

$$\begin{aligned}
{}_nV_x^{(m)} &= A_{x+n} - P_x^{(m)}\,\text{a}_{x+n}^{(m)} \\
&= A_{x+n} - \left[P_x + \frac{m-1}{2m}P_x^{(m)}(P_x+d)\right]\left[\text{a}_{x+n} - \frac{m-1}{2m}\right] \\
&= A_{x+n} - P_x\cdot\text{a}_{x+n} - \frac{m-1}{2m}P_x^{(m)}(P_x+d)\text{a}_{x+n} + P_x^{(m)}\cdot\frac{m-1}{2m} \\
&= {}_nV_x - \frac{m-1}{2m}P_x^{(m)}\cdot(1-{}_nV_x) + P_x^{(m)}\cdot\frac{m-1}{2m} \\
&= {}_nV_x\left[1+\frac{m-1}{2m}P_x^{(m)}\right].
\end{aligned}$$

Again, $\quad {}^{(n)}_{\ t}V_x^{(m)} = A_{x+t}^{(n)} - {}^{(n)}P_x^{(m)}\,\text{a}_{x+t}^{(m)}$

where $\quad A_{x+t}^{(n)} \times (1+i)^{\frac{1}{2n}} = \bar{A}_{x+t} = A_{x+t}(1+i)^{\frac{1}{2}}$ approximately,

$$\text{a}_{x+t}^{(m)} = \text{a}_{x+t} - \frac{m-1}{2m} \text{ approximately,}$$

and hence $\quad {}^{(n)}P_x^{(m)} = \dfrac{A_x^{(n)}}{\text{a}_x^{(m)}} = \dfrac{\left(1+\frac{n-1}{2n}i\right)A_x}{\text{a}_x - \frac{m-1}{2m}}$ approximately.

9. For a whole life policy with instalment premiums,

$${}_nV_x^{[m]} = \left(1-\frac{m-1}{2m}P_x^{[m]}\right)A_{x+n} - P_x^{[m]}\cdot\text{a}_{x+n}^{(m)} \text{ approximately,}$$

But $\quad P_x^{[m]} = P_x + \dfrac{m-1}{2m}P_x^{[m]}\cdot d$ approximately

therefore ${}_nV_x^{[m]} = {}_nV_x$, which should be obvious if n is an integer.

10. For a joint life assurance with annual premiums we have

$$_{n}V_{xy}=A_{x+n:y+n}-P_{xy}\,.\,a_{x+n:y+n}\,.$$

For a survivorship assurance with annual premiums payable during the duration of the contract we have, if both lives have survived n years,

$$_{n}V_{\overline{xy}}=A_{\overline{x+n:y+n}}-P_{\overline{xy}}\,a_{\overline{x+n:y+n}}\,.$$

But if x alone has died within the n years, the reserve at the end of n years, is $A_{y+n}-P_{\overline{xy}}\,a_{y+n}$.

CHAPTER IX

Select and Aggregate Tables

1. Although we have referred to different mortality tables which may have been made for each sex or for different races or classes, we have regarded the rate of mortality in each table as being wholly dependent upon the age attained. A table formed on this assumption is called an aggregate table. Tables produced from population data are aggregate tables.

2. In dealing with assured lives or annuitants we have to contend with an additional factor called "selection". Selection may be briefly defined as the attempt by one party to a contract to protect himself against an undue advantage which might under certain circumstances be realized by the other party.

As examples:

(1) applicants for life assurance or assurance against disability are generally subjected to a medical examination by an assurance company before a policy is issued.

(2) applicants for assurance without medical examination are limited in the amount of assurance which may be taken out, say $5,000 at one time, and satisfactory evidence of general insurability is demanded.

(3) evidence of selection has been found in the choice of the type of assurance taken out. Other conditions being equal, lives who take out endowment or limited payment life policies have been shown to exhibit lighter mortality rates at the outset than those who choose whole life or term policies.

(4) the person who contemplates purchasing an immediate annuity on his own life would only do so provided he felt that his health was very good.

3. Select tables have been deduced to exhibit the mortality after the act of selection in such cases where sufficient data can be obtained. Select tables are really a set of tables—one for each age at entry.

The amount of labour involved in the formation of all the elementary functions in a complete select table would be prohibitive. If for example the ages at entry were from 15 to 60 there would be 46 separate tables. It is however generally assumed that the effect of selection "wears off" in a definite number of years called the period of selection; so that, after this period has passed the mortality may be taken as a function of the attained age regardless of the duration. The result is that a "trunk" or "ultimate" table is formed into which run the different branches covering the period of selection for the different ages at entry.

4. The English Life Table No. 8 is an aggregate table formed from population statistics. So is the Carlisle Table. The H^M Table is an aggregate table formed from assured male lives. This table is based upon the experience up to 1863 of some twenty Assurance companies in Great Britain. The American Experience Table is an ultimate table mainly based upon the experience of one of the oldest American companies. It was published about 1868.

The $O^{[NM]}$ is a select table based on assured male lives who had taken out whole-life, non-participating assurances in English companies. The experience covered the period 1863-1893. The period of selection was taken as 5 years. The ages at entry for the select portion of the table were from 20 to 75 inclusive.

5. In order to distinguish the age at entry in our symbols, a bracket [ ] is written around this age in the suffix. The duration is then added until the ultimate part of the table is reached when a single suffix can be used.

Thus $q_{[x]}$ represents the rate of mortality between ages x and $x+1$ for a person selected at age x.

$q_{[25]+3}$ represents the rate of mortality for a person who is aged 28, having been selected at age 25.

If the period of selection is 5 years, then

$q_{[25]+n}=q_{25+n}$ where $n \geqq 5$, and this symbol represents the rate of mortality for a person who is aged $25+n$ and who had been selected at an age younger than $21+n$.

6. From the experience which is to be used as the basis of a select table with a five year selection period, the values of q_x are calculated for the ultimate table by excluding from the experience the first five years of each life after the presumed selection. Then using only those lives which were selected at, say age 25, the values of $q_{[25]}$, $q_{[25]+1}$ . . . . . . $q_{[25]+4}$ are calculated and graduated to fit in with the ultimate q_{30}, q_{31} . . . , When this has been done for each age at entry we have complete tables of $q_{[x]+n}$ for all values of x and for values of n from 0 up to 4.

If 20 is the youngest age at entry, we first form the ultimate table by using the ultimate values of q_x on the ultimate value of

$$l_{25} = l_{[20]} \times p_{[20]} \times p_{[20]+1} \times p_{[20]+2} \times p_{[20]+3} \times p_{[20]+4}$$

where $l_{[20]}$ is an arbitrarily chosen number.

Other select values of $l_{[x]+n}$ are formed as follows:

$$l_{[x]+4} = l_{x+5} \times p_{[x]+4}^{-1}$$
$$l_{[x]+3} = l_{[x]+4} \times p_{[x]+3}^{-1}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Referring to the $O^{[NM]}$ Table of the numbers living, we find for example that there are 84,924 lives in the ultimate table living at age 40 out of 87,402 who were living at age 37 two years after selection, or out of 88,536 who were select lives at age 35, or out of 89,187 who were living at age 35 and were select lives before age 31.

The student should examine the $O^{[NM]}$ table for himself and verify the deduction of the $l_{[x]+n}$ values.

CHAPTER X

Office Premiums

1. So far we have been dealing with the pure theory of the value of payments contingent upon the survival or failure of human life. Such payments are however generally made under contracts issued by companies which deal in assurances and annuities. These companies cannot be operated without expense, and so the premiums paid must be "loaded" to cover at least all necessary expenses.

Generally the "net premium" for assurances is calculated at a rate of interest lower than the rate which the company may reasonably expect to earn, also at rates of mortality higher than those which the policy holders will exhibit. The net premium so calculated is then further loaded for expenses with a margin for safety. Periodically return payments or credits are made to policy holders out of the excess interest earned, the saving due to favourable mortality, and any unexpended portion of the additional loading after expenses have been met. These returns are called "Profits" or "Surplus" or "Dividends". The loading for policies which will participate in profits is usually in the form of a percentage addition to the net premium plus a constant addition per $1,000. of assurance; for example, if the net premium was $20.16 per $1,000, and the loading formula was 20% plus $3.00 per $1,000, the office premium for a $5,000 policy would be $\$100.80 \times 1.20 + \$15.00 = \$135.96$.

An old English form of loading on the annual net premium for a whole life policy provided for

(1) An initial commission of 1% on the sum assured spread over the life.

(2) An annual constant of $\frac{1}{8}$ of 1% on the sum assured.

(3) An annual charge equivalent to $7\frac{1}{2}\%$ of the net premium plus the loadings (1) and (2).

The office premium would be

$$\mathbf{P}_x = 1.075\left(P_x + \frac{.01}{\mathrm{a}_x} + .00125\right)$$

$$= P_x\ (1.086) + \text{a constant depending on the rate of interest.}$$

The usual formula for whole life policies is

$$\mathbf{P}_x = P_x(1+k)+c = (P_x+c')(1+k)$$

where k, c and c' are constants.

2. Assuming that the expenses of a limited payment or increasing premium policy for a whole life assurance are about the same as for an ordinary whole life policy we have for a limited payment policy

$${}_n\mathbf{P}_x \cdot \mathrm{a}_{x\,\overline{n}|} = \mathbf{P}_x \cdot \mathrm{a}_x$$

and for a policy under which the premium is to be $r\pi_x$ for n years followed by π_x for the balance of life

$$r\pi_x \cdot \mathrm{a}_{x\,\overline{n}|} + \pi_x(\mathrm{a}_x - \mathrm{a}_{x\,\overline{n}|}) = \mathbf{P}_x \cdot \mathrm{a}_x.$$

3. If the expenses of a twenty pay life policy are as follows:

(1) an initial commission of 60% of the first premium

(2) initial expenses amounting to \$10 per \$1,000 assured

(3) annual expenses amounting to

(*a*) $7\frac{1}{2}$% of the office premium during the premium paying period, and

(*b*) $\frac{3}{4}$ of 1% of the office premium thereafter

(4) settlement expenses amounting to \$7.50 per \$1,000 assured.

The value of the receipts by the office is ${}_{20}\mathbf{P}_x \cdot \mathrm{a}_{x\,\overline{20}|}$ where ${}_{20}\mathbf{P}_x$ is the office premium.

The value of the payments to be made by the office is

$$A_x(1.0075) + .6\times{}_{20}\mathbf{P}_x + .01 + .075\times{}_{20}\mathbf{P}_x \cdot \mathrm{a}_{x\,\overline{20}|}$$

$$+ .0075\times{}_{20}\mathbf{P}_x(\mathrm{a}_x - \mathrm{a}_{x\,\overline{20}|})$$

therefore

$${}_{20}\mathbf{P}_x = \frac{.01+(1.0075)A_x}{.9325\times \mathrm{a}_{x\,\overline{20}|} - .0075\times \mathrm{a}_x - .6}.$$

The office premiums for Endowment policies or for long term temporary assurances may be similarly calculated but the office premiums for short term policies are affected by other than mathematical considerations.

4. In actuarial work the experience of the past is used as a guide to the future. There are many different types of lives but the past experience available is limited. It would be folly, apart from the labour, to break the past experience up into a multitude of parts so that we could have a separate table for each type of life. "Standard" tables are formed from the past experience representing the mortality likely to be experienced in the near future by more or less well defined groups of persons. When an individual is to be considered whose expected mortality differs greatly from the average, an estimated adjustment is made to the results which can be obtained from the standard table so as to fit his case.

5. Sometimes the conditions under which the applicant is living or the medical examination and record of family history demanded before a policy is issued may reveal risks or defects indicating for example:

(1) that the proposed life is subject to more than the normal risk of death during the next n years,

(2) that the proposed life while quite healthy at present comes of a stock which is not long lived,

(3) that the proposed life exhibits general weaknesses greater than is normally the case at his age.

The first case might not be accepted for a term policy but might be accepted for a whole life policy with a reduced amount at risk during the first n years.

The second case could be accepted for a term policy or an endowment policy or even for a whole life policy under which the amount at risk would diminish during the later years.

The third case, if accepted, might be "rated up", *i.e.*, regarded as an older life than he actually is.

The first and second cases, when quite definite, present no mathematical difficulty.

The third case may be illustrated as follows:

A man who is 30 years of age asks for a whole life policy but has been "rated up" to age 45 by the medical examiner. He objects to paying the premium for age 45, asking to be allowed to pay the premium for age 30 and agreeing that only one half of the sum assured shall be payable in the event of death during the first n years.

To find n, we have

$$P_{30} \, . \, a_{45} = A_{45} - \tfrac{1}{2} A^{1}_{45\,\overline{n|}} \, .$$

If we have a table of $A^{1}_{x\,\overline{n|}}$ we can enter it inversely opposite $x = 45$ with the value $2[A_{45} - P_{30} \, . \, a_{45}]$ to find n. Or, using commutation values, we have

$$\frac{M_{30}}{\mathbf{N}_{30}} \cdot \frac{\mathbf{N}_{45}}{D_{45}} = \frac{M_{45} + M_{45+n}}{2D_{45}}$$

therefore $$M_{45+n} = 2\frac{M_{30} \times \mathbf{N}_{45}}{\mathbf{N}_{30}} - M_{45}$$

from which n can be found.

Where offices are protected by agreements among themselves against rate-cutting almost any method which is agreed upon may be used in calculating the office premiums so long as the resulting premiums are abundantly safe. But where competition is allowed free play office premiums should be calculated from select rather than from aggregate or ultimate tables, and the rate of interest used may be close to the rate which the office expects to earn.

CHAPTER XI

EXAMPLES

1. Given that $q_x = \dfrac{x-45}{100}$ for values of x from 90 to 100 inclusive, and that $l_{90}=1{,}000$, form the l and d columns starting at age 90. What stationary population can be supported by 1,000 annual entrants aged 90?

2. If $d_x=k+cx$ and $l_{100}=d_{100}=1$, find l_{10} and ${}_{10}p_{10}$ in terms of either k or c.

3. If $l_x=ks^xw^{x^2}g^{c^x}$ for all values of the variable x obtain an expression for the force of mortality.

4. Assuming a uniform distribution of deaths prove that

$$({}_{\frac{1}{2}}p_x-p_x)+({}_{\frac{3}{2}}p_x-{}_{2}p_x)+({}_{\frac{5}{2}}p_x-{}_{3}p_x)+\text{etc.}=\tfrac{1}{2}.$$

5. There are three lives (x), (y) and (z). You are given that $q_x=.02$, $q_y=.025$, and $q_z=.03$. Find the probabilities of all the different possibilities that may happen as regards death and survival during the next year. Find the sum of these probabilities.

6. Given ${}_{10}p_{30}=.920$, ${}_{10}p_{40}=.890$ and $p_{40}=.991$, calculate the probability that a person aged 30 will die (i) before age 50, (ii) between ages 40 and 50, (iii) in his 41st year.

7. The probability that a man aged 20 and another man aged 40 will both be alive at the end of 20 years is .4. Out of 96,000 men alive at age 20, 6,000 will die before they attain age 30. Find the probability that a man now aged 30 will die within the next 30 years.

8. Three men, A, B and C, are all of the same age x. Express the following probabilities:

(i) That A alone will be living at age $x+n$.

(ii) That only one of them will be dead n years hence.

(iii) That they will die in alphabetical order.

(iv) That A will die after B but during the life of C.

9. There are three lives, (x), (y), (z). Write down expressions for the following probabilities:

(i) That they will not all be dead n years hence.

(ii) That they will not all be living n years hence.

(iii) That (x) and at least one other will be living n years hence.

(iv) That they will all live ten years and that (x) will die after (y) but before (z).

(v) That (x) will die in the first year and be the first to die.

10. The probability that A will die within ten years is .2. The probability that A, B and C will all be alive ten years hence is .42. The probability that at least one of the three will be alive ten years hence is .985. Find the probability that A and B alone will be living at the end of the tenth year.

11. Given $|_{n}Q_{x:x+n}=.9$ and ${}_{n}p_{x}=.5$, find the probability that of three lives each aged x

(a) at least two will die between ages $x+n$ and $x+2n$

(b) the third death will take place between the ages $x+n$ and $x+2n$.

12. Express Q^{2}_{xyz}, Q^{3}_{xyz}, $Q^{1}_{x\overline{yz}}$, $Q^{1}_{\overline{xy}:z}$ in terms of probabilities determining at the first death.

13. Differentiate ${}_{n}p_{x}$ (1) with regard to x

(2) with regard to n.

14. Prove $\dfrac{e_{x}.e_{x+1}.e_{x+2}\;.\;.\;.\;.\;.\;.\;.\;.\;.\;e_{x+n-1}}{(1+e_{x+1})(1+e_{x+2})\;.\;.\;(1+e_{x+n})}={}_{n}p_{x}.$

15. Show that

$$1+e_x=q_x+p_x(1+q_{x+1})+{}_2p_x(1+q_{x+2})+\text{etc.}$$

16. If $\mu_x=\dfrac{1}{100-x}$, find l_x, and the complete expectation of life.

17. Obtain an expression for the average age at death of the present members aged x and upwards in a stationary community supported by births alone and unaffected by migration.

If the law of mortality is as indicated by the following values of d_x—the limiting age of the table being 100—find the average age at death of the present members aged 94 and upwards.

$$d_{94}=11,\ d_{95}=9,\ d_{96}=7 \text{ and so on.}$$

18. What do the following integrals represent?

(i) $\displaystyle\int_0^n t\,.\,l_{x+t}\cdot\mu_{x+t}\cdot dt$

(ii) $\displaystyle\int_n^{n+m} {}_tp_x\cdot\mu_{x+t}\cdot dt$

(iii) $\displaystyle\int_0^\infty ({}_tp_y+{}_tp_z-{}_tp_{yz})\cdot{}_tp_x\cdot\mu_{x+t}\cdot dt.$

19. Given $A_{25}=.356$, $p_{25}=.992$, and $i=.03$, find A_{26}.

20. Prove $A_x=i\sum\limits_{n=1}^{\infty}a_{\overline{n|}}\,.\,{}_np_x-d\sum\limits_{n=1}^{\infty}\left[(1+i)v^n.\dfrac{(1+i)^n-1}{i}.{}_{n-1}p_x\right]+1.$

21. Give a formula for the single premium at age x for a whole life assurance, the sum assured during the first year to be (1.015), during the second year $(1.015)^2$, during the third year $(1.015)^3$ and so on.

22. Prove $\displaystyle\int_x^{x+1}(\mu_x+\delta)dx=\text{colog }vp_x.$

23. Prove $A_{\overline{xxx}}=A_{xxx}-3(A_{xx}-A_x).$

24. Show that

$$A_{\overline{xyz}} = A_{\overline{xyz}} = A_x + A_{\overline{yz}} - A_{x\,\overline{yz}} = A_x + A_y + A_z - A_{xy} - A_{yz} - A_{xz} + A_{xyz}$$

25. Examine the following statements critically:

(i) $1 = ia_{x\overline{n}|} + (1+i)A_{x\overline{n}|}$

(ii) $a_x - a_{x\overline{n}|} = v^n a_{x+n}$

(iii) $1 = da_{x\overline{n}|} + A^{1}_{x\overline{n}|} + v^n_n p_x.$

Explain any corrections that may be necessary.

26. Explain (i) $A_x + iA_x - q_x(1 - A_{x+1}) = A_{x+1}$

(ii) $a_x + ia_x - p_x(1 + a_{x+1}) = 0.$

27. Show that

$$\mathrm{a}_x = \sum_{n=1}^{\infty} \left[v^n (1+i) \frac{(1+i)^n - 1}{i} ({}_{n-1}p_x - {}_n p_x) \right].$$

28. Prove that $a^{(2)}_{x\overline{n}|} = \frac{1}{4}(3a_{x\overline{n}|} + \mathrm{a}_{x\overline{n}|})$ approximately.

29. Show that (i) $a_{\overline{xy}:\overline{zw}} = a_{x\,\overline{zw}} + a_{y\,\overline{zw}} - a_{xy\,\overline{zw}}$

(ii) $a_{\overline{xyzw}} = a_{\overline{\overline{xy}\,\overline{zw}}} = a_{\overline{xy}} + a_{\overline{zw}} - a_{\overline{xy}\,\overline{zw}}$

$$= \Sigma a_x - \Sigma a_{xy} + \Sigma a_{xyz} - a_{xyzw}.$$

30. Find the H^M. 3% value of

(i) An annuity payable quarterly for not more than ten years to a life now aged 60. What correction would you suggest to make the annuity apportionable?

(ii) An annuity payable monthly, first payment at age 70, to a life now aged 50. The rent to be \$100 a month from age 70 to age 80 and \$125 a month thereafter.

31. The value at 3% of an apportionable annuity payable yearly is 12.100. Find the value of an annuity on the same life payable quarterly but not apportionable.

32. Identify the two expressions for the value of a reversionary annuity to (x) after the death of (y)

$$\sum_{n=1}^{\infty} v^n \left({}_np_x - {}_np_{xy}\right) \quad \text{and} \quad \sum_{n=1}^{\infty} v^n \frac{l_{x+n} \cdot d_{y+n-1}}{l_x \cdot l_y} a_{x+n}.$$

Discuss these expressions when the mortality and/or interest rates used while the annuity is in reversion differ from those used while the annuity is in possession.

33. Explain in words and express in terms of single and joint life annuities

$$a_{\overline{xy}\,\overline{n}|}, \quad a_{x|\overline{n}|}, \quad a_{xy|z}, \quad a_{\overline{xy}|z}, \quad a_{x|yz}, \quad a_{x|\overline{yz}}.$$

34. Prove (i) $\dfrac{d}{di} a_x = -v(Ia)_x$

(ii) $\dfrac{d}{di} A_x = -v(IA)_x$.

35. If $l_x = a - bx$, show that $\bar{a}_x = \bar{a}_\infty - \mu_x (I\bar{a})_\infty$.

36. Prove $(a_x - a_{xy}) : (A_y - A_{\overline{xy}}) :: (1+i) : i$.

37. Given that $a_{28:\overline{29}|} = 15.200$ and $a_{28:\overline{30}|} = 15.412$ at 4%, find the values of $A_{28:\overline{30}|}$, $A^{1}_{28:\overline{30}|}$ and $A_{28:\overline{30}|}^{\ \ 1}$.

38. Given that $A_x = .29095$ and $i = .035$, find a_x and P_x.

39. If $a_x = 11.649$ and $P_x = .04525$, find A_x and the rate of interest.

40. Given that $A_x = .3801$ and $P_x = .02109$ find a_x and the rate of interest.

41. Prove (i) $P^1_{x\overline{t}|} = v - a_{x\overline{t}|}\ (P_{x\overline{t}|} + d)$

(ii) $A_{x\overline{n}|} = \dfrac{a_\infty - a_{x\overline{n-1}|}}{1 + a_\infty}$

and interpret in words.

42. Given $P_{35\,\overline{20}|}=.04012$, ${}_{20}P_{35}=.02740$ and $A_{55}=.56615$, find the value of $P^{\;1}_{35\,\overline{20}|}$.

43. Express ${}_tP_{x\,\overline{n}|}$ in terms of $P_{x\,\overline{t}|}$, $P_{x\,\overline{n}|}$ and the rate of discount.

44. Given $P_{\overline{xy}\,\overline{n}|}=.0379$, $P_{\overline{xy}}=.0083$ and $i=.04$, find the value of ${}_nP_{\overline{xy}}$.

45. What do the following integrals represent?

(i) $\int_0^\infty t\,.\,v^t\,.\,{}_tp_x\,.\,\mu_{x+t}\,.\,dt$

(ii) $\int_0^\infty v^t\,.\,(1-{}_tp_y)\,{}_tp_x\,.\,dt$

(iii) $\int_0^\infty v^t\,.\,{}_tp_{xy}\,.\,\mu_{y+t}\,\bar{a}_{x+t}\,.\,dt.$

46. A whole life policy is taken on the condition that only half the annual premium shall be payable for the first five years. Under these conditions the premium is .04. If $P_x=d=.035$, find $a_{x\,\overline{4}|}$.

47. Prove and interpret

$$P_x^{[m]}=P_x+\frac{m-1}{2m}\,.\,P_x^{[m]}\,.\,d.$$

48. Express the value of the following benefits:

(i) A unit payable 12 years after the death of (x).

(ii) A unit payable 12 years hence provided that (x) has died in the interval.

(iii) A pension of one per annum to (x), first payment at age $x+n$.

Find the annual premium limited to 10 years for each of these benefits. Assume that $n>10$ in (iii).

49. In return for a yearly premium of $\dfrac{v^{20}}{\mathrm{a}_{x\,\overline{20}|}}$ an insurance company offers to a man aged x, a guarantee of 1 payable at the end of

20 years if he be alive, and a sum S payable 15 years after his death if he should die within the 20 years. Find S.

50. A man aged 35 wants a whole-life policy, but wants it issued from age 25 so that he will pay the premium for age 25. He also offers to pay up the back premiums from age 25 with interest.

Another man, also aged 35, wants to purchase an annuity on his life beginning at age 65 by premiums payable up to age 65. He also wants the contract dated back to the time when he was aged 25 and offers to pay the back premiums with interest.

Discuss the equity of these suggestions.

51. A man aged x offers to pay a lump sum of $P_{x-n:\overline{2n}|}\ (s_{\overline{n+1}|}-1)$ and an annual premium of $P_{x-n:\overline{2n}|}$ for an endowment policy maturing at age $x+n$. Find the amount of the policy.

A man aged x offers a single premium of $(a_{x-n}-a_{x-n:\overline{2n}|})(1+i)^n$ for an annuity, first payment at age $x+n$. Find the amount of the annuity.

52. The annual premium for a whole life policy is .04 for five years and .02 thereafter or .0475 for five years and .0175 thereafter. Find the values of P_x and ${}_5P_x$.

53. The annual premium for an endowment assurance is 2.727 for the first five years and 7.46 thereafter or 1.361 for the first five years and 8.46 thereafter. Find the uniform annual premium.

54. For what benefit is $\dfrac{P_{x\overline{n}|}}{d}$ the single premium? Express your answer in words.

55. If \$25 is the annual premium payable for 10 years for a certain benefit on the death of (27), what should be the quarterly premium payable for the life of (27) to secure the same benefit using H^M 3% values?

56. Given a complete table of $a_{x\overline{n}|}$ for all values of x and n, how would you find $A_{x\overline{n}|}^{\;\;1}$? If you were also given the rate of interest how would you find $A^{1}_{x\overline{n}|}$ and $A_{x\overline{n}|}$?

57. Prove
(i) $dN_{x-1}=D_x-M_x$
(ii) $dN_x=vD_x-M_x$
(iii) $D_{x+1}+M_x=vD_x+M_{x+1}$
(iv) $N_{x-1}=(1+i)\,(M_x+N_x)$.

58. Differentiate (i) with respect to x,
(ii) with respect to i,
$\bar{a}_x,\ \bar{A}_x,\ \bar{P}_x,\ \bar{D}_x,\ \bar{M}_x$.

59. Differentiate (i) with respect to x
(ii) with respect to i
(iii) with respect to n
$\bar{a}_{x\overline{n}|},\quad \bar{A}^{1}_{x\overline{n}|},\quad A_{x\overline{n}|}^{\;\;1}$.

60. Explain in words

$$(P_{x\overline{n}|}-P_x)\frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}=1-\frac{A_{x+n}-A_x}{1-A_x}.$$

61. If a population consists of l_x persons assumed to be at exact age x, l_{x+1} at age $x+1$, and so on down to the extremity of life, show that the total fund to be raised to provide an assurance of unity on the death of each person is equal to $l_x(a_x+A_x)$.

62. Show that $\mathrm{a}_x=d(I\mathrm{a})_x+(IA)_x$.

63. On the death of each member of a society of n members all now aged x, each surviving member is to pay \$100. Find the present value of these payments.

64. Calculate the annual level premium payable for 5 years in respect of a life aged 35 for a two year temporary assurance to

commence at age 38. The sum assured for the first year of risk is \$10,000, and for the second year of risk is \$5,000. Use H^M 4% values.

65. A is aged 25 and B is aged 35. They wish to contribute equally to some charity. A promises to pay \$100 a year on each of his birthdays from the 26th to the 35th inclusive. B agrees to pay x dollars if and when he attains age 45. Find x by the use of H^M 3% values.

66. A agrees to pay B \$1,000 at his (A's) death. B agrees to pay A \$19 a year so long as A lives. What should be the age of A to make this equitable on the basis of 3% interest and H^M mortality

(i) if B pays his \$19 in advance each year?

(ii) if B pays it in arrears each year?

67. Show how to apply a given sum B on a whole-life policy issued n years ago so as to

(*a*) limit the number of future premiums

(*b*) convert the policy into an endowment assurance.

68. An assurance policy provides in return for an annual premium limited to 20 payments

(*a*) on death within 25 years \$5,000 plus a bonus of \$250 for each premium paid,

(*b*) \$5,000 on survival at the end of 25 years, and

(*c*) \$5,000 at death if after 25 years.

Find the net annual premium for a man now aged 30 on the H^M $3\frac{1}{2}$% basis.

69. Find in commutation symbols an expression for a_{25} where the interest rate is to be $4\frac{1}{2}$% for the first five years, 4% for the next 10 years, decreasing thereafter by $\frac{1}{2}$% at the end of each 10 year period until a minimum rate of $2\frac{1}{2}$% is reached.

70. Find an expression for the net purchase price of a complete annuity of \$1,000 per annum payable quarterly, with the condition that if death occurs in the first policy year four-fifths of the net

price will be refunded; this death benefit to decrease by one-fifth of the net purchase price for the second, third and fourth policy years.

71. A loan of $a_{\overline{n}|}$ to (x) is to be repaid by n equal annual amounts of 1 each, including principal and interest. Assuming the same rate of interest throughout, prove that the single premium for an assurance to secure the amount outstanding should (x) die during the n years is $a_{\overline{n}|} - a_{x\overline{n}|}$.

72. Prove that the single premium for an assurance of 1 on a life aged x with return of the premium along with the sum assured is equal to the annual premium to secure a perpetuity of 1 per annum of which the first payment is to be made at the end of the policy year in which (x) shall die.

73. A man aged 30 has contracted to pay an annuity-certain of \$500 for five years, first payment at the end of 25 years, the liability falling on his estate if he should die. Find the net annual premium limited to 25 payments required to relieve his estate of this obligation in the event of his death. Use H^M $3\frac{1}{2}\%$ values.

74. (*a*) Prove that

$$\frac{1}{s_{\overline{m}|}\,D_x}\left[s_{\overline{1}|}\,C_x + s_{\overline{2}|}\,C_{x+1} + \;.\;.\;.\; + s_{\overline{m}|}\,C_{x+m-1}\right] = \frac{v\mathrm{a}_{x\overline{m}|}}{s_{\overline{m}|}} - \frac{D_{x+m}}{D_x}.$$

(*b*) Demonstrate that the annual premium for a whole life assurance during each of the first m years is composed of (1) the savings fund premium necessary to accumulate the mth terminal reserve and (2) the level annual premium for an m-year term assurance of such nature that the amount of assurance payable in any year is the amount by which the accumulation of the savings fund premium falls short of the full amount of the policy.

75. On the net premium basis throughout, find the annual premium for a whole life policy to (30) with premiums limited to 20 years, provided that each premium to be paid is to be less than

the first premium by 5% on all premiums already paid. Use H^M $3\frac{1}{2}\%$ values.

76. Find an H^M 5% approximate present value of the payments of an annuity-due of \$150 a year which are to accumulate during the life of (40) at 1% per annum and to be paid in one sum at the end of the year of his death.

77. If the single premiums for a pure endowment of \$100 payable on (x) surviving n years are \$49 and \$41, with and without return of premium in event of death before age $x+n$, what should be the single premium if only one-half of it were so returnable?

78. Derive the formula for calculating the return premium factor by which a twenty-payment life premium should be multiplied to provide for the return of all premiums paid in event of the death of the assured within twenty years.

79. Find on the H^M 3% basis

(*a*) the net single premium for an assurance payable if (20) reaches age 60, the premium to be returned without interest in the event of death before attaining that age,

(*b*) the net annual premium for a 20-year joint life endowment assurance on two persons each aged 40. Compare your result with $2\,P_{40\,\overline{20}|} - P_{\overline{20}|}$.

80. A man aged 35 wishes a whole life policy by annual premiums for such an amount as will provide \$10 per month for 20 years certain, to be entered on from the date of death. He also wishes a provision incorporated to the effect that if his wife, aged 30, survive him by 20 years the sum of \$10 per month shall continue to be payable for the remainder of her lifetime. The premium for the added clause to his policy is only to be payable during the joint lifetime of the pair. Find expressions for the premiums.

81. Prove that $\dfrac{A_x}{(IA)_x}$ is the annual premium for a whole life assurance on (x) subject to the condition that compound interest,

at the rate used in the calculation, for each year on the premiums paid up to and including the year of death is to be paid together with the sum assured.

82. Prove that $\frac{1-da_x}{a_x-d\,(Ia)_x}$ is the annual premium for a whole life assurance on (x) subject to the condition that interest, at the rate used in the calculation, on all premiums paid is to be paid to the policy holder or his estate at the end of each year up to and including the year of death.

83. Find in terms of commutation symbols the annual premium in each of the two preceding questions and prove that they are identical.

84. Ground rents of \$1,000 a year are payable for 20 more years, one payment having just been made. An hospital is to receive these rents after the death of a man now aged 60. Find the value of the interest of the hospital on the basis of H^M mortality and 3% interest.

85. Use the commutation symbols to prove

(i) $\frac{A_{x+n}-A_x}{1-A_x}+\frac{1+a_{x+n}}{1+a_x}=1$

(ii) ${}_tP_x-P^{\,1}_{x\overline{n}|}=P_{x\overline{n}|}^{\;\;\,1}\cdot{}_{n:t}V_x.$

86. If $1-A_{x+2n}=A_{x+2n}-A_{x+n}=A_{x+n}-A_x$, express ${}_nV_{x+n}$ in terms of ${}_nV_x$ and find the numerical values of ${}_nV_{x+n}$ and ${}_nV_x$.

87. Given ${}_{10}V_{25}=.09894$, $P_{25}=.01611$, $q_{35}=.009$ and $i=.03$, find ${}_{11}V_{25}$.

88. Prove that ${}_nV_x=1-(1-{}_1V_x)\,(1-{}_1V_{x+1})\;.\;.\;.\;(1-{}_1V_{x+n-1})$.

89. Prove that $P_{x+n}(1-{}_nV_x)=P_x+d\,.\,{}_nV_x$ and explain the relation in words. Write down similar relations, n being less than 20, which will exist for (i) a twenty year endowment policy

(ii) a twenty pay life policy.

90. Prove ${}_nV_x = \dfrac{P_x - P^{1}_{x\overline{n}|}}{P_{x\overline{n}|}^{\ \ 1}}$ and explain in words.

91. Derive the following and give an interpretation in words of each

(a) $${}_nV_x = \left(P_x + \frac{P_x}{i} + 1\right)A_{x+n} - \left(P_x + \frac{P_x}{i}\right)$$

(b) $$P_x = \frac{|_nA_{x+m} + v^n \cdot {}_np_{x+m} \cdot {}_{n+m}V_x - {}_mV_x}{1 + |_{n-1}a_{x+m}}.$$

92. Prove that

(i) $D_x = M_x + d(D_x + N_x)$

(ii) $$\frac{dA_x}{1 - A_{x\overline{n}|}} - P_x = P_{x\overline{n}|}^{\ \ 1}\,(A_{x+n} - {}_nV_x).$$

93. Prove the following, through commutation symbols or otherwise

(i) $A_{x+n} = {}_nV_x(1 - A_x) + A_x$

(ii) $$a_{x\overline{n}|} = \frac{1 - A_{x\overline{n+1}|}}{d} - 1.$$

Explain the truth of each formula in words.

94. Under a certain mortality table $x = 100A_x$ for all values of x when the rate of discount is 4%. Find expressions for a_x and P_x. Also find the reserve after 15 years on a whole life policy issued at age 35.

95. Give two different expressions for the reserve after (i) ten, (ii) twenty years under a combination policy taken out at age 35 to provide \$10,000 in the event of death before age 50 and a pension of \$1,000 a year, first payment at age 50. The premiums are to be paid for 15 years only. Calculate the net premium on the H^M 3% basis.

96. Prove that the value of a policy with an annual premium of π and which has been exactly n years in force may be written

$$V=A_{x+n}-\pi \mathrm{a}_{x+n}+\frac{(\pi-P_x)\mathrm{N}_x}{D_{x+n}}$$

where π is the annual premium for

(*a*) a limited premium whole life policy.

(*b*) an endowment policy.

(*c*) a term policy.

Will the formula hold for a pure endowment or for a deferred assurance?

97. Find the H^M 3% annual premium, limited to 20 payments, at age 25 for a policy under which in the event of death before age 35 only the premiums paid are to be returned, but if (25) attain age 35 the premium is to be cut in half and the benefit is to be $10,000 payable at the end of the year of death.

Find both prospectively and retrospectively the reserve under this policy at the end of the (i) 7th year, (ii) 27th year.

98. Find the reserve after n years under:

(i) A t-payment life policy where $t<n$.

(ii) A t-year endowment policy where $t>n$.

(iii) A policy payable only in the event of death between ages $x+t$ and $x+t+s$ when $n<t$ and also when $t<n<t+s$.

In each case the policy is taken out at age x and premiums are payable for t years only.

99. A policy under which $2,000 will be paid on death before age 60, $500 at age 60 if the policy holder attains that age, and $1,000 on death after age 60, is taken out at age 30. Find the annual premium limited to 20 payments. Also find the reserve by both retrospective and prospective methods (i) after 10 years (ii) after 35 years.

100. If for two mortality tables $\mathrm{a}_x=(1+k)\mathrm{a}'_x$ at rate i for a certain range of consecutive values of x, where k is a constant, show

that the relation $q'_x = q_x + \frac{k}{va_{x+1}}$ will hold for the same range, and conversely.

101. A man aged 60 has a whole life policy for $1,000 on which the 31st annual premium is just due. He cannot pay it, so he suggests that the company shall continue his assurance until the reserve is exhausted. Use H^M 3% values to find how many more years his assurance will last.

102. Given $P_x = .01240$ and $a_{x+n} = 13.900$ at 4% interest, what paid-up policy could be given in respect of a whole life policy effected at age x which has been n years in force?

103. A life assured, now aged 40, under a Non-Profit Endowment Assurance Policy for $5,000 issued ten years ago and maturing 20 years hence, desires that it be converted into a Non-Profit Endowment Assurance Policy to mature ten years hence.

(*a*) What would be the amount of the increased premium if the sum assured were to remain at $5,000?

(*b*) What would be the amount of the policy if it were desired to reduce the sum assured in lieu of increasing the premium?

(*c*) If the sum assured for the term part of the contract were to remain at $5,000, what would be the amount of the pure endowment that could be offered in lieu of increasing the premium?

Use H^M $3\frac{1}{2}$% values.

104. A thirty-year endowment policy, with premiums reducing at the end of twenty years, is to be issued with the provision that the reserve at the end of twenty years is to be equal to the sum of the premiums paid. Give expressions for the annual premium to be paid during the first twenty years and for the annual premium to be paid thereafter.

105. A man aged 50 takes out a 5-year endowment assurance for $10,000 by paying annual premiums. Make up a schedule showing to the nearest dollar for each year the interest earned, the cost of the assurance, and the resulting reserve. Use 3% H^M values.

106. A man aged 35 effects a policy on the life of his son aged 5, payable at death after age 21. The net premiums are to be returned if the son dies before age 21; also, should the father die before the son attains age 21, all premiums are to cease until the son reaches age 25, thereafter to resume and continue payable during the life of the son. Obtain a formula for the net premium, and show the value of the policy at the end of 10 years.

107. A man aged 50 has a whole life policy on which the 25th premium is just due. He wishes to change it into an endowment policy maturing at age 60.

(i) If the premium is to be unchanged find the amount of the new policy.

(ii) If the amount of the policy is to be unchanged find the new premium.

(iii) If he demands half the reserve in cash and can pay no more premiums, what free policy can be given to him?

108. Ten years ago A, then aged 27, effected by annual premium a policy for \$10,000 payable at his death. He now wishes the policy converted, subject to the same premium, into an endowment assurance payable by annual instalments of \$1,000 commencing at age 60 or at his previous death. On the H^M $3\frac{1}{2}\%$ basis how many annual instalments should he receive?

109. A whole life assurance is effected by annual premium on a life aged x whose health is impaired and who may be assumed to be subject to the same rates of mortality as a normal life aged $x+n$. Instead of charging an extra premium the office agrees to accept the normal premium for age x provided the sum assured is temporarily reduced for 10 years, the reduction being $10k$ for the first year and decreasing by k at the end of each year. Find k in terms of x and n, also give expressions for the policy value after 5 years by the retrospective and prospective methods and show that the expressions are equal.

110. A whole life assurance for \$1,000 without profits is effected on a life aged 30 who is subject to the rates of mortality of a special

table under which the $3\frac{1}{4}\%$ net annual premium at age 30 is $\frac{1}{4}\%$ greater than the H^M 3% net premium. If the policy values on the special basis at $3\frac{1}{4}\%$ are equal to the H^M 3% values throughout life, find the special rate of mortality at age 40.

111. A pure endowment for $1,000 on the attainment of age 50 was effected by annual premiums at age 30 with return of premiums in event of previous death. After 5 years it is desired to cancel the remaining premiums by a single payment which will be returnable in event of death before maturity. If all the office premiums are calculated on the H^M 3% basis with a loading of 10% find the required commutation price.

112. Find two expressions for the net reserve at age 25 in the case of an annual premium policy securing to a child aged 10 an assurance payable at age 60 or at its previous death if after age 25 with a return of all premiums paid in event of death before age 25, assuming $\mathbf{P}=(P+c)(1+k)$.

113. If $\pi_{x\overline{t}|}$ is the net premium for an annuity of 1 per annum deferred t years with the return of the net premiums in the event of death before the annuity commences, prove that the value of the contract at the end of n years $(n<t)$ on the same basis as that on which the premium is calculated is

$$n\pi_{x\overline{t}|} \cdot A^{\;1}_{x+n:\overline{t-n}|} + \left(1 - \frac{\pi_{x\overline{t}|}}{\pi_{x+n:\overline{t-n}|}}\right) \cdot {}_{t-n|}a_{x+n}.$$

114. A thirty-five year pure endowment is taken at age x under the following conditions: no return of any kind if assured die in the first five years, return of net premiums without interest if death occur in the next 20 years, return of gross premiums with compound interest at rate j if assured die in the last 10 years. Find the net annual premium at interest rate i where the gross annual premium is obtained therefrom by loading with a percentage and a constant.

115. Find a formula for the office premium for a twenty-pay-life policy for $1,000, allowing for expenses as follows: 80% of the first

premium, 10% of all subsequent premiums, and .5% of the sum assured each year after all premiums have been paid.

116. Construct a formula for the office premium for a twenty-pay-life policy so as to allow for

(i) initial expenses amounting to 75% of the first premium

(ii) annual expenses amounting to 10% of all premiums including the first

(iii) annual expenses, equivalent to 5% of the office premium, for every year of life after the premium period has elapsed

(iv) settlement expenses amounting to 1% on the sum assured.

117. Find the office premium payable for 20 years at age 30 for a pension of $1,000 a year, first payment at age 60. The expenses are 60% of the first premium, 5% of all subsequent premiums, $5 a year after the premium period and before age 60 and 1% of the pension payment each year so long as it is payable. Use H^M 3% values.

118. Find an expression for the office premium on a whole life policy taken out at age x and subject to the condition that in the event of a claim before age $x+n$, only the premiums paid will be returned. Allow for a loading of 20%, plus $3.00 per thousand and assume the office premium is less than $\frac{1}{n}$.

119. An assurance on (x) is payable at the end of twenty years if (x) be then alive or at his death should this occur during the last ten years of this period. If (x) dies during the first ten years, the gross premiums paid are to be returned. Assuming that the gross premium is obtained by adding a percentage and a constant to the net premium find expressions for the gross annual premium payable for fifteen years only, and also for the sixth year policy value, using both prospective and retrospective methods.

120. A loan of $10,000, at 7% interest, is made to a man whose life can be assured at a premium of $30 per thousand. What

annual sum should the borrower pay in advance each year so long as he may live in order that the lender may have his interest in full without fear of loss of capital and also that no liability will rest upon the estate of the borrower?

121. Calculate on the H^M table at 3% interest the value of a life annuity payable yearly to a person aged 50 subject to the condition that the payments shall continue in any event until the total sum paid shall amount to the purchase price.

122. A, whose life can be assured at a premium of $30.00 per $1,000, borrows $8,000 and agrees to pay for it principal and interest by annual payments of $850 a year during his life. What rate of interest does this give the lender if A pays the $850 in (i) advance, (ii) arrears each year?

123. The yearly rents of an estate belong in equal shares to four sisters, being equally divided at the end of each year amongst all who survive that year, the last survivor taking the whole for the rest of her life. Find an expression for the value of the interest of any one of them.

124. Determine the respective interests of (x), (y) and (z) in an annuity payable annually until the second death, if it is to be equally divided between (x) and (y) during their joint lives and after the death of either to be divided in like proportions between (z) and the survivor during what may remain of their joint lives.

125. An annuity payable until the death of the last survivor of (x), (y) and (z), is to be divided equally between (x) and (y) during their joint lives. If (x) dies first, (y) and (z) are to enjoy it equally during the balance of their joint lives and the survivor of them is to have the whole; but if (y) dies first, (x) is to enjoy the whole during the balance of his life and after his decease the whole annuity goes to (z) if he be living. Find the value of their respective interests.

126. Given 1,000 couples, all husbands aged x and all wives aged y, write down expressions for the following:

(*a*) the number of widows living 10 years hence

(*b*) the number of couples that will be broken during the tenth year

(*c*) the probability that exactly one couple will be broken during the first year

(*d*) the probability that at least 2 couples will be broken within 10 years

(*e*) the annual premium to be paid by each married man for a reversionary annuity of 1 per annum to his wife after his death.

127. Two brothers (x) and (y) desire to provide an income of $1,000 annually for themselves and their two sisters (w) and (z); this income to begin at the death of the first brother and to continue until the death of the last survivor of the remaining three lives. Deduce an expression for the value of this benefit in terms of single and joint life annuities.

128. An office has on its books 100 whole life policies on lives now aged 50, one half of the policies being for $1,000 and the remainder for $2,000. On the H^M basis what is the probability that (*a*) three deaths will occur in the next year? (*b*) $2,000 will be payable in claims in the next year?

129. If the "select period" is five years, express in terms of the "number living"

(i) ${}_{2|3}Q_{[30]}$

(ii) the probability that a person who entered two years ago at age 30 will die within a year after attaining age 35.

130. Find in select commutation symbols the net annual premium to provide for a child now aged 6 a temporary annuity of $1,000 a year for four years, first payment at age eighteen. In the event of the death of the child before the age of eighteen, the net premiums will be returned without interest.

131. An office has ten policies of \$1,000 each assured at age 40. Given $q_{40}=.02$, find the probability that the claims will not exceed \$5,000 in the next year.

132. Show that

$$A_{xy\overline{n}|} = A_{x\overline{n}|} + A_{y\overline{n}|} - A_{\overline{n}|} - d(a_{\overline{n}|} - a_{\overline{xy}\,\overline{n}|}),$$ and explain its meaning.